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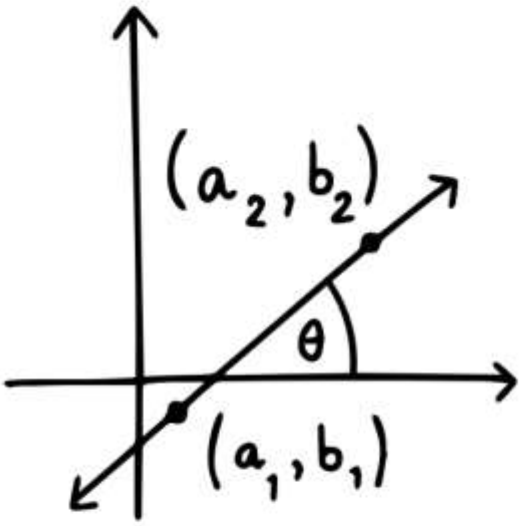
Matrix Algebra



$y = mx$

$A_x = B_y + C = 0$

$$a^2 + b^2 = c^2$$

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By the end of this chapter you should be familiar with:

- Matrix, its order and elements
- Algebra of matrices
- Determinant
- Inverse
- Linear System
- Eigenvectors and Eigenvalues
- Diagonalization
- Transformations
- Applications

MATRIX DEFINITIONS AND OPERATIONS

Matrix refers to an ordered **rectangular arrangement of numbers** which are either real or complex or functions. We enclose Matrix by [ ] or ( ) where $a_{i,j}$ represents the element located in row i and column j.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Different types of matrices are:

- **Row matrix** – A row matrix has only one row but any numberof columns. A matrix is said to be a row matrix if it has only one row. For example, A = [−1/2 √5 2 3] is a row matrix of order 1 × 4. In general, $A = [a_{ij}]_{1 \times n}$ is a row matrix of order 1 × n.
- **Column matrix** – A column matrix has only one column but any number of rows. A matrix is said to be a column matrix if it has only one column. For example, A =[3 3 1] is a column matrix of order 3 × 1. In general, $B = [b_{ij}]_{m \times 1}$ is a column matrix of order m × 1.
- **Square matrix** – A square matrix has the number of columns equal to the number of rows. A matrix in which the number of rows is equal to the number of columns is said to be a square Thus, an m × n matrix is said to be a square matrix if m = n and is known as a square matrix

of order 'n'. For example, $A = \begin{pmatrix} 1 & 6 & 7 \\ 2 & 5 & 8 \\ 3 & 4 & 9 \end{pmatrix}$ is a square matrix of order 3. In general, A = [a_{ij}] m × m is a square matrix of order m. And if it doesn't have equal number of rows and columns then it is called a rectangular matrix.

- **Diagonal matrix** – A square matrix B = [bij] m × m is said to be a diagonal matrixif all its non-diagonal elements are zero, that is a matrix B =[b_{ij}]$_{m\times m}$ is said to be a diagonal matrix if b_{ij} = 0, when i ≠ j. For example, A = [1 0 0 1] where A is also known as an identity matrix as it has all 1's in its diagonal.
- **Zero Matrix** – A matrix is said to be zero matrix or null matrix if all its elements are zero.
  For Example, A = [0] where A is of order 1.
- **Upper Triangular matrix** – A square matrix in which all the elements below the diagonal are zero is known as the upper triangular matrix. For example,
  $$A = \begin{pmatrix} 1 & 6 & 7 \\ 0 & 5 & 8 \\ 0 & 0 & 9 \end{pmatrix}$$
- **Lower Triangular matrix** – A square matrix in which all the elements above the diagonal are zero is known as the upper triangular matrix. For example, $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ 3 & 4 & 9 \end{pmatrix}$

ADDING AND SUBTRACTING MATRICES

The **addition** of two matrices A_{m*n} and B_{m*n} gives a matrix C_{m*n}. The elements of C is the sum of corresponding elements in A and B which can be shown as

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 12 & 14 \end{bmatrix}$$

The **subtraction** of two matrices A_{m*n} and B_{m*n} gives a matrix C_{m*n}. The elements of C are difference of corresponding elements in A and B which can be represented as:

$$\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

Addition of matrices is **commutative** which means A+B = B+A **Addition** of matrices is **associative** which means A+(B+C) = (A+B)+C whereas **subtraction** is **neither**. The matrices must have the **same order** to be computed.

SCALAR MULTIPLICATION

Scalar multiplication refers to the **product of a real number** and a **matrix**. In scalar multiplication, each entry in the matrix is multiplied by the given scalar.

Example: A = [1 2 0 1]

then 2A = [2 4 0 2]

MULTIPLACTION OF MATRICES

The **multiplication** of two matrices A_{m*n} and B_{n*p} gives a matrix C_{m*p}. It means number of columns in A must be equal to number of rows in B to calculate C=A*B. To calculate element c11, multiply elements of 1st row of A with 1st column of B and add them (5*1 + 6*4) which can be shown as:

Multiplication of matrices is **non-commutative** which means A*B ≠ B*A **Multiplication** of matrices is **associative** which means A*(B*C) = (A*B)*C

But, A × I = I × A = A where **I** is **the identity matrix**.

Example: From the given data find the final grades of the following students

Student	Tests	Projects	Homework	Quizzes
Alexandra	**92**	**100**	**89**	**80**
Megan	**72**	**85**	**80**	**75**
Brittney	**88**	**78**	**85**	**92**

Type	Weight
Tests	**40%**
Projects	**15%**
Homework	**25%**
Quizzes	**20%**

$$\text{Final grades} = \begin{pmatrix} 92 & 100 & 89 & 80 \\ 72 & 85 & 80 & 75 \\ 88 & 78 & 85 & 92 \end{pmatrix} \begin{pmatrix} 0.4 \\ 0.15 \\ 0.25 \\ 0.2 \end{pmatrix} = \begin{bmatrix} 90 \\ 77 \\ 87 \end{bmatrix}$$

Solution:

Alexandra has a 90, Megan has a 77, and Brittney has an 87.

DETEREMINANT AND INVERSES

The **determinant** of the 2 × 2 matrix A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the number ad – cb. The above sentence is abbreviated as **det(A) = ad – cb**

If the **determinant** of the matrix **0**, then the **inverse** of the matrix **exists**.

For a 2 × 2 matrix A = the **inverse** is given as: $\mathbf{A}^{-1} = \dfrac{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}{\det(A)}$ For a 3 × 3 matrix, We can find A^{-1} in such a way that $AA^{-1} = I$ or we can use the calculator.

For determinant we can use the given formula:

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \cdot \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \cdot \det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \cdot \det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

Note: $(PQ)^{-1} = Q^{-1}P^{-1}\ P^{-1}Q^{-1}$

LINEAR SYSTEM

A given set of equations can be written in matrix form given by: **AX = B** where **A** contains the **matrix of coefficients of x and y**. **X** contains a **matrix x and y** and **B** contains the **matrix of the constants**.

Example: Solve the system of equations using matrices. 2x + 3y = 3 and x – y =4

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}^{-1}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \frac{-1}{5}\begin{pmatrix} -1 & -3 \\ -1 & 2 \end{pmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Solution:

Therefore, x = 3 and y = -1.

There is a solution to the system when A is non-singular. If B = 0, the system is homogeneous. A homogeneous system will always have a solution, called the trivial solution, X = 0 when A is non-singular. When A is singular then the system has infinitely many solutions.

AREA OF TRIANGLE

We can find the area of the triangle using determinants.

Example: Find the area of the triangle ABC whose vertices are A(1,3) , B(5,-1) and C(-2,5)

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 \\ 5 & -1 & 1 \\ -2 & 5 & 1 \end{vmatrix} = -4$$

Solution:

Area = |1/2 × determinant value| = 2

LINES

For lines in plane,

If $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \neq 0$ then the **points cannot lie on a line** but if the **determinant value is 0** then the **points lie on the line**.

To find the **two-point equation** of a line we use: $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$

CRYPTOGRAPHY

The **process of encryption** is known as **cryptography**. Here, we are going to use **Hill Cypher**.

Lets take an example to understand the working.

For ease, we will use a 2×2 key matrix for the encryption. The message we want to encrypt is “UTES”. Since our key matrix is 2×2, we must split the message into chunks of two letters. This is true for all matrices, so for an NxN matrix, the message would be split into chunks of N letters. Each letter in the message is assigned a numerical value, ranging from 0 up to the modular base. For our encryption, we will use the letters A-Z (so the modulus is 26). While in practice the letters can be assigned any value (as long as the sender and recipient both know the corresponding values), we will use the straightforward approach. So, A = 0, B = 1, C = 2, etc. all the way to Z = 25. Below we see how our message is assigned:

$UTES$ = (20 19) , (4 18)

Our key matrix K for this example will be: K = (4 1 3 7)

Note that the key matrix must be known by the receiver in order to decode the message.

Now for each chunk of our message, we perform a matrix multiplication against the key matrix.

$$\begin{pmatrix} 4 & 1 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} 20 \\ 19 \end{pmatrix} = \begin{pmatrix} 99 \\ 193 \end{pmatrix} \text{ } Mod\ 26 = \begin{pmatrix} 21 \\ 11 \end{pmatrix} = \begin{pmatrix} V \\ L \end{pmatrix}$$

$$\begin{pmatrix} 4 & 1 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} 4 \\ 18 \end{pmatrix} = \begin{pmatrix} 34 \\ 138 \end{pmatrix} \text{ } Mod\ 26 = \begin{pmatrix} 8 \\ 8 \end{pmatrix} = \begin{pmatrix} I \\ I \end{pmatrix}$$

Our encrypted message is VLII.

For decrypting this we find K^{-1} and multiply it with (21 11) and (8 8) to get back our original matrices which says UTES.

GAUSS-JORDON ELIMINATION

Gauss-Jordan Elimination is an algorithm that can be used to solve systems of linear equations and to find the inverse of any invertible matrix. It relies upon three **elementary row operations** one can use on a matrix:

1. Swap the positions of two of the rows
2. Multiply one of the rows by a nonzero scalar.

3. Add or subtract the scalar multiple of one row to another row.

Example: Use Gauss-Jordan elimination to solve the system: x + 3y + 2z = 2, 2x + 7y + 7z = -1 and 2x + 5y + 2z = 7

Solution: The augmented matrix of the system is

$$R2 \to R2 - 2R1 \text{ and } R3 \to R3 - 2R1 = \begin{pmatrix} 1 & 3 & 2 & : & 2 \\ 0 & 1 & 3 & : & -5 \\ 0 & -1 & -2 & : & 3 \end{pmatrix}$$

$$R2 \to R2 - (-R3) = \begin{pmatrix} 1 & 3 & 2 & : & 2 \\ 0 & 1 & 3 & : & -5 \\ 0 & 0 & 1 & : & -2 \end{pmatrix}$$

$$R1 \to R1 - 2R3 \text{ and } R2 \to R2 - 3R1 = \begin{pmatrix} 1 & 3 & 0 & : & 6 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & -2 \end{pmatrix}$$

$$R1 \to R1 - 3R2 = \begin{pmatrix} 1 & 0 & 0 & : & 3 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & -2 \end{pmatrix}$$

Therefore, x = 3, y = 1 and z = -2

ROW REDUCED ECHELON FORM

The purpose of **Gauss-Jordan Elimination** is to use the three elementary row operations to convert a matrix into reduced-row echelon form. A matrix is in **reduced-row echelon form**, also known as **row canonical form**, if the following conditions are satisfied:

1. All rows with only zero entries are at the bottom of the matrix
2. The first nonzero entry in a row, called the **leading entry** or the **pivot**, of each nonzero row is to the right of the leading entry of the row above it.
3. The leading entry, also known as the pivot, in any nonzero row is 1.
4. All other entries in the column containing a leading 1 are zeroes.

In the previous example, the row reduced echelon form was

$$= \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

EIGENVECTORS AND EIGENVALUES

An **eigenvector** of a matrix A is a vector whose product when multiplied by the matrix is a scalar multiple of itself. The corresponding multiplier is often denoted as and referred to as an **eigenvalue**.

$\mathbf{A}\cdot\mathbf{v} = \lambda\cdot\mathbf{v}$

In this equation **A** is an n-by-n matrix, **v** is a non-zero n-by-1 vector and λ is a scalar (which may be either real or complex). Any value of λ for which this equation has a solution is known as an eigenvalue of the matrix **A**. It is sometimes also called the characteristic value. The vector, **v**, which corresponds to this value is called an eigenvector. The eigenvalue problem can be rewritten as

$\mathbf{A}\cdot\mathbf{v} - \lambda\cdot\mathbf{v}=0$

$\mathbf{A}\cdot\mathbf{v} - \lambda\cdot\mathbf{I}\cdot\mathbf{v}=0$

$(\mathbf{A} - \lambda\cdot\mathbf{I})\cdot\mathbf{v}=0$

If **v** is non-zero, this equation will only have a solution if

$|\mathbf{A} - \lambda\cdot\mathbf{I}|=0$

This equation is called the characteristic equation of **A**, and is an n^{th} order polynomial in λ with n roots. These roots are called the eigenvalues of **A**. We will only deal with the case of n distinct roots, though they may be repeated. For each eigenvalue there will be an eigenvector for which the eigenvalue equation is true.

Example: Find eigenvalues and eigenvectors of $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$

$$\left|\mathbf{A}-\lambda\cdot\mathbf{I}\right| = \left|\begin{bmatrix}0 & 1\\-2 & -3\end{bmatrix}-\begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix}-\lambda & 1\\-2 & -3-\lambda\end{bmatrix}\right| = \lambda^2+3\lambda+2=0$$

Solution:

and the two eigenvalues are $\lambda_1=-1$, $\lambda_2=-2$

All that's left is to find the two eigenvectors. Let's find the eigenvector, $\mathbf{v}_1$, associated with the eigenvalue, $\lambda_1=-1$, first.

$$\mathbf{A}\cdot\mathbf{v}_1=\lambda_1\cdot\mathbf{v}_1$$

$$\left(\mathbf{A}-\lambda_1\right)\cdot\mathbf{v}_1=0$$

$$\begin{bmatrix}-\lambda_1 & 1\\-2 & -3-\lambda_1\end{bmatrix}\cdot\mathbf{v}_1=0$$

$$\begin{bmatrix}1 & 1\\-2 & -2\end{bmatrix}\cdot\mathbf{v}_1=\begin{bmatrix}1 & 1\\-2 & -2\end{bmatrix}\cdot\begin{bmatrix}v_{1,1}\\v_{1,2}\end{bmatrix}=0$$

$$v_{1,1}+v_{1,2}=0,\quad \text{so}$$

$$v_{1,1}=-v_{1,2}$$

$$-2\cdot v_{1,1}+-2\cdot v_{1,2}=0,\quad \text{so again}$$

$$v_{1,1}=-v_{1,2}$$

$$\mathbf{v}_1=k_1\begin{bmatrix}+1\\-1\end{bmatrix}$$

where k_1 is an arbitrary constant. Note that we didn't have to use +1 and -1, we could have used any two quantities of equal magnitude and opposite sign.

Going through the same procedure for the second eigenvalue:

$$\mathbf{A} \cdot \mathbf{v}_2 = \lambda_2 \cdot \mathbf{v}_2$$

$$(\mathbf{A} - \lambda_2) \cdot \mathbf{v}_2 = \begin{bmatrix} -\lambda_2 & 1 \\ -2 & -3-\lambda_2 \end{bmatrix} \cdot \mathbf{v}_2 = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} v_{2,1} \\ v_{2,2} \end{bmatrix} = 0 \quad \text{so}$$

$$2 \cdot v_{2,1} + 1 \cdot v_{2,2} = 0 \quad (\text{or from bottom line: } -2 \cdot v_{2,1} - 1 \cdot v_{2,2} = 0)$$

$$2 \cdot v_{2,1} = -v_{2,2}$$

$$\mathbf{v}_2 = k_2 \begin{bmatrix} +1 \\ -2 \end{bmatrix}$$

DIAGONALIZATION

An n × n matrix A is **diagonalizable** if it is similar to a diagonal matrix: that is, if there exists an invertible n × n matrix P and a diagonal matrix D such that

$A = PDP^{-1}$

$A^n = PD^nP^{-1}$

where **D** is a **diagonal matrix** containing the **eigenvalues** and **P** is a **matrix containing eigenvectors**.

We can use the above example and prove that PDP^{-1} would give us A back.

MATRICES AND GEOMETRIC TRANSFORMATIONS

The method we can use to find the matrix representing a transformation is given by the Matrix Basis theorem.

Let **T** be a **transformation** represented by a **matrix M**:

T: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} a \\ c \end{pmatrix}$ and T: $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} b \\ d \end{pmatrix}$ then M = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

1. **Reflection in the x-axis:** $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

   M = $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

2. **Reflection in the y-axis:** $\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}0\\1\end{pmatrix}$
   M = $\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$

3. **Reflection in y = x:** $\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}1\\0\end{pmatrix}$
   M = $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$

4. **Reflection in y = -x:** $\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}-1\\0\end{pmatrix}$
   M = $\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$

5. **Horizontal dilation by a constant k (stretch or shrink):** $\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}k\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}0\\1\end{pmatrix}$
   M = $\begin{pmatrix}k & 0\\0 & 1\end{pmatrix}$

6. **Vertical dilation by a constant m (stretch or shrink):** $\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}0\\k\end{pmatrix}$
   M = $\begin{pmatrix}1 & 0\\0 & m\end{pmatrix}$

7. The matrix **M =** $\begin{pmatrix}\boldsymbol{k} & \mathbf{0}\\\mathbf{0} & \boldsymbol{m}\end{pmatrix}$ represents a **horizontal dilation of magnitude k and a vertical dilation of constant m**. When **k = m** this is called **scaling**.

AFFINE TRANSFORMATIONS

In order to incorporate the idea that both the basis and the origin can change, we augment the linear space u, v with an origin o. Note that while u and v are basis vectors, the origin o is a point. We call u, v, and t (basis and origin) a frame for an affine space. Then, we can represent a change of frame as:

p' = x.u + y.v + o

The two types of special affine transformations are rotations and translations.

To **rotate** use the matrix $\begin{pmatrix}\boldsymbol{cos\theta} & \boldsymbol{-sin\theta}\\\boldsymbol{sin\theta} & \boldsymbol{cos\theta}\end{pmatrix}$

A **translation** of **h units horizontally** and **k units vertically** is achieved by: $\begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+h \\ y+k \\ 1 \end{pmatrix}$

Example: The square PQRS with vertices P(-2,4), Q(-6,-3), R(1,-7), S(5,0) undergoes a transformation described by $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Determine the coordinates of the vertices of the image PQRS and graph it.

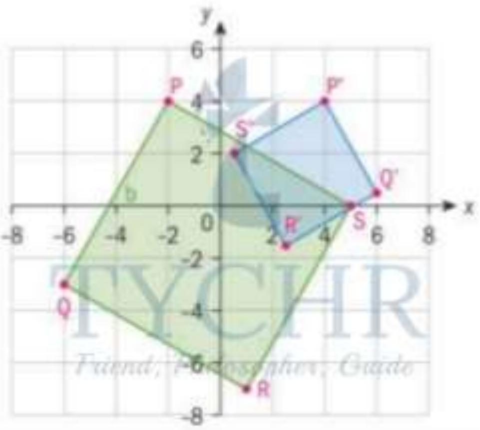
Solution:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} -2 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$(-6, -3) \rightarrow (6, 0.5)$

$(1, -7) \rightarrow (2.5, -1.5)$

$(5, 0) \rightarrow (0.5, 2)$



If two transformation are S and T then S o T is known as a composition of two transformations.

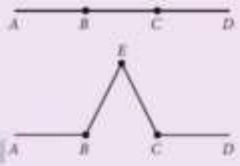
FRACTALS

The 8 transformation equations used to generate this image are:

1. $T_1\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}\frac{1}{3}&0\\0&\frac{1}{3}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}$
2. $T_2\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}\frac{1}{3}&0\\0&\frac{1}{3}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}\frac{1}{3}\\0\end{pmatrix}$
3. $T_3\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}\frac{1}{3}&0\\0&\frac{1}{3}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}\frac{2}{3}\\0\end{pmatrix}$
4. $T_4\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}\frac{1}{3}&0\\0&\frac{1}{3}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}\frac{2}{3}\\\frac{1}{3}\end{pmatrix}$
5. $T_5\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}\frac{1}{3}&0\\0&\frac{1}{3}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}\frac{2}{3}\\\frac{2}{3}\end{pmatrix}$
6. $T_6\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}\frac{1}{3}&0\\0&\frac{1}{3}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}\frac{1}{3}\\\frac{2}{3}\end{pmatrix}$
7. $T_7\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}\frac{1}{3}&0\\0&\frac{1}{3}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\\frac{2}{3}\end{pmatrix}$
8. $T_8\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}\frac{1}{3}&0\\0&\frac{1}{3}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\\frac{1}{3}\end{pmatrix}$

Example:

The **Koch curve** is constructed by removing the middle third of a line segment (say length 1 unit) and replacing it with two sides of an equilateral triangle. Here is the first iteration.



The Koch curve can be created with four transformations:

1. The first shrinks the whole segment to one third of its original size. This maps AD to AB.
2. The second is a dilation of ratio 1:3 followed by a rotation of 60° and a horizontal translation of 1:3. This maps AD to BE.
3. The third transformation is a dilation of ratio 1:3 followed by a rotation of 260° and a translation of $\frac{1}{2}$ in the horizontal direction and $\frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{3}}{6}$ in the vertical direction. This maps AD to EC.
4. The fourth transformation is a dilation of ratio 1:3 followed by a translation of 2:3 in the horizontal direction. This maps AD to CD.

(a) Find the matrices representing each transformation.

(b) Draw the first three iterations.

Solution:

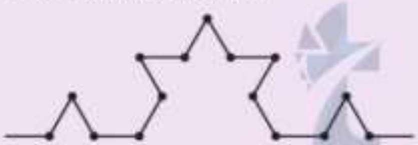
1. $\begin{pmatrix}1/3&0\\0&1/3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}$

2. $\begin{pmatrix}cos60&-sin60\\sin60&cos60\end{pmatrix}\begin{pmatrix}1/3&0\\0&1/3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}1/3\\0\end{pmatrix}=\begin{pmatrix}1/6&-1/\sqrt{2}\\1/\sqrt{2}&1/6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}1/3\\0\end{pmatrix}$

3. $\begin{pmatrix}\cos(-60)&-\sin(-60)\\\sin(-60)&\cos(-60)\end{pmatrix}\begin{pmatrix}1/3&0\\0&1/3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}1/2\\1/\sqrt{2}\end{pmatrix}=\begin{pmatrix}1/6&1/\sqrt{2}\\-1/\sqrt{2}&1/6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}1/2\\1/\sqrt{2}\end{pmatrix}$

4. $\begin{pmatrix}1/3&0\\0&1/3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}2/3\\0\end{pmatrix}$

For the second iteration, we apply the same procedure on each segment created in the first iteration:



For the third iteration, we apply the same procedure on each segment created in the second iteration:

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