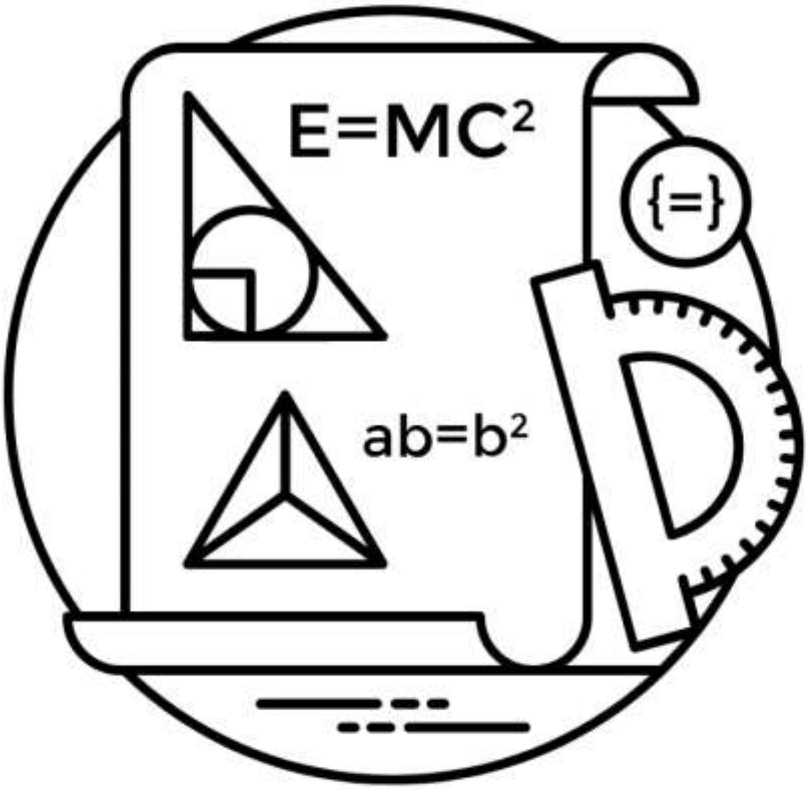
TYCHR

Friend, Philosopher, Guide

Integral Calculus



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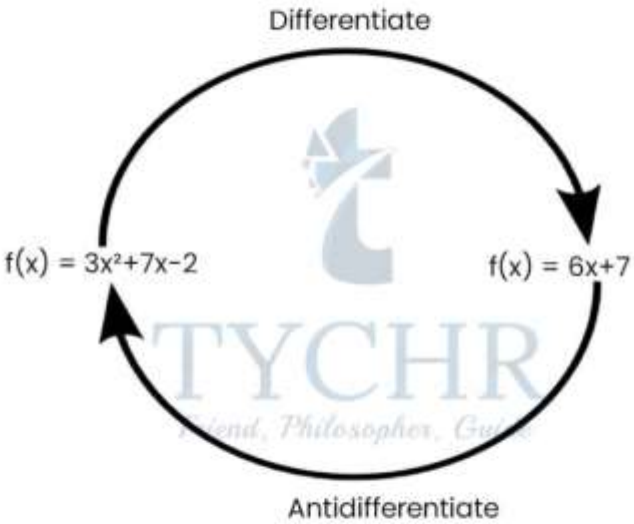
By the end of this chapter, you will be familiar with:

- Integration as antidifferentiation
- Definite integrals
- Fundamental theorem of calculus
- Area under curves
- Boundary conditions
- Integration of various functions
- Integration by substitution
- Kinematics using integration

ANTIDERIVATIVE AND INDEFINITE INTEGRAL

By now we are familiar with differentiating common functions.
In this unit we carry out the process of differentiation in reverse. That is, we start with a given function, f(x) say, and ask what function or functions, F(x), would have f(x) as their derivative. This leads us to the concepts of an antiderivative and integration.
Consider the function $f(x)=3x^2+7x-2$. Suppose we write its derivative as f(x), that is $f(x) = \frac{dF}{dx}$. We already know how to find this derivative by differentiating tem by term to obtain $f(x) = \frac{dF}{dx} = 6x + 7$ Suppose now that we work back to front and ask ourselves which function or functions could possibly have 6x+7 as a derivative. Clearly, one answer to this question is the function $3x^2+7x-2$. We say $f(x)=3x^2+7x-2$ is an **antiderivative** of f(x)=6x+7.



There are however other functions which have derivative 6x+7. Example
$3x^2+7x+3$
$3x^2+7x$
$3x^2+7x-12$

The reason why all of these functions have the same derivative is that the constant term disappears during differentiation. So, all of these are antiderivatives of 6x+7.
Given any antiderivative of f(x), all others can be obtained by simply adding a different constant. In other words, if F(x) is an antiderivative of f(x), then so too is f(x)+C for any constant C.
If f'(x)=f(x) we write f(x)dx=f(x)+C

The expression ∫f(x)dx is called an **indefinite integral.**
Table of some common indefinite integrals:

F(X)	∫F(X)Dx=F(X)+C
k (constant)	kx+C
xn	$\frac{x^{n+1}}{n+1} + C$
sin x	−cos x +C
cos x	sin x +C
ex	ex+C
$\frac{1}{x}$	ln x +C

Rules for integration:

1. Constant multiple rule $\int kf(x)dx = k\int f(x)dx$
2. Sum or difference rule $\int[f(x)\pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

Ex. Find indefinite integrals of the following:

1. $\int 8dx$
2. $\int \frac{2}{x^5} dx$
3. $\int e^x + 4\sqrt{x}$

Solution:

1. $\int \frac{2}{x^5} dx = 8(x) + C = 8x + C$
2. $\int \frac{2}{x^5} dx = 2\left(\frac{x^{-5+1}}{-5+1}\right) + C = -\frac{1}{2}x^{-4} + C$
3. $\int e^x + 4\sqrt{x} = \int e^x + \int 4\sqrt{x} = e^x + 4\frac{(x^{\frac{3}{2}})}{\frac{3}{2}} + C = e^x + \frac{8}{3}x^{\frac{3}{2}} + C$

INTEGRATION BY SUBSTITUTION

Integration by Substitution (also called "u-Substitution" or "The Reverse Chain Rule") is a method to find an integral, but only when it can be set up in a special way.

Ex. $\int \sin sin\,(3x + 2)dx$ We know that $\int \sin sin\,x\,dx = -\cos cos\,x + C$

But here, we have another function inside the sine function.

Recall that, in chain rule of differentiation, we differentiate the inner function after differentiating the outer.

Let us multiply and divide the differentiation of the inner function in the intergral. $= \frac{1}{3}\int 3 \sin sin\,(3x + 2)dx$ Now, we will put 3x+2=u

Differentiate the above equation:

3dx=du

Substitue the value of 3x+2 and 3dx in the integral $= \frac{1}{3}\int \sin sin\,u\,du$

Now, we can easily integrate- $= \frac{1}{3}(-\cos cos\,u + C)$

Putting the value of u: $= \frac{1}{3}(-\cos cos\,(3x + 2) + C)$

Using this simple substitution method, we have the following results:

f(x)	∫f(x)
(ax+b)n	$\frac{1}{a(n+1)}(ax+b)^{n}+C$
eax+b	$\frac{1}{a}e^{ax+b}+C$
$\frac{1}{ax+b}$+b	$\frac{1}{a}\ln ln\|ax+b\| + C$ Taking more examples:

Ex. $\int(5x^{3}+2)^{8}x^{2}dx$ Let u=5x3+2

hence, du=15x2dx

We will need a factor of 15 in the integral

$=\frac{1}{15}\int(5x^{3}+2)^{8}15x^{2}dx$ Substituting the value of u and du:

$=\frac{1}{15}\int u^{8}du$ Now we can simply integrate:

$=\frac{1}{15}\left(\frac{u^{9}}{9}\right)+C$ Substituing the value of u: $=\frac{1}{135}(5x^{3}+2)^{9}+C$

Ex. $\int xsin^{3}(2x^{2})\cos\cos\left(2x^{2}\right)dx$ Let sin(2x2)=u

hence, cos (2x2) 4xdx=du

We will need a factor of 4 in the integral

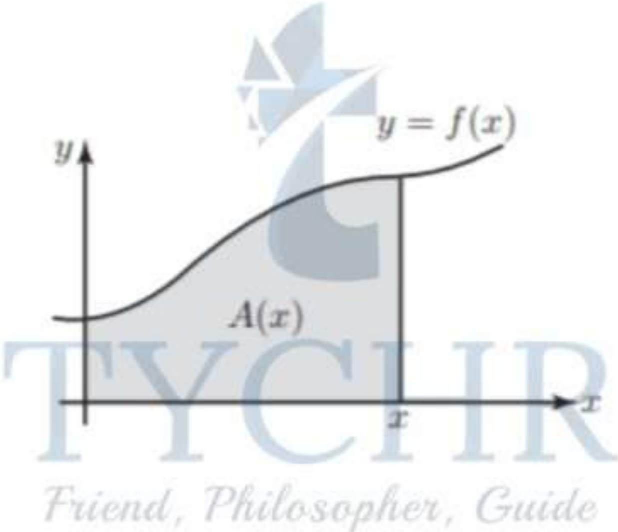
$=\frac{1}{4}\int 4xsin^{3}(2x^{2})\cos\cos\left(2x^{2}\right)dx$ Substituting the value of u and du: $=\frac{1}{4}\int u^{3}du$ Now we can simply integrate: $=\frac{\frac{1}{4}\left(u^{4}\right)}{4}+C=\frac{1}{16}u^{4}+C$ Substituing the value of u:

$=\frac{\left(\sin\sin\left(2x^{2}\right)\right)^{4}}{16}+C$

AREA UNDER CURVE AND DEFINITE INTEGRAL

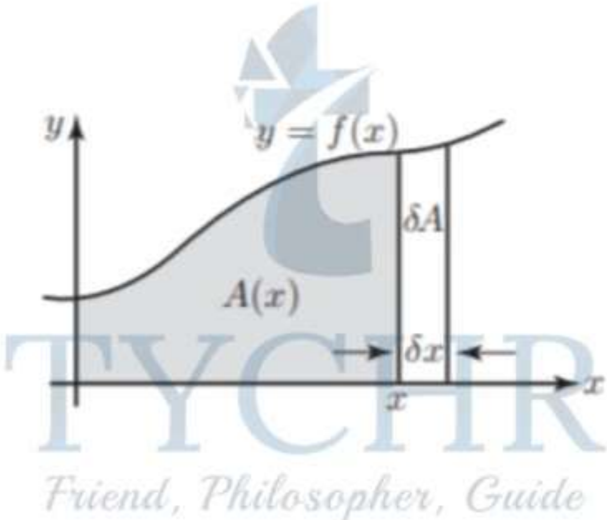
Consider that portion of the graph of $y=f(x)$.

- Suppose $f(x)$ lies entirely above the x axis. Clearly the area bounded by the graph A, and the x axis will depend on upon the value



  of x, $A=A(x)$
- Suppose we consider an additional contribution made to this area by moving a little to the right.
- Let this additional contribution be denoted by δA. Then δA is the change in area produced by increasing x by δx.

- Note then that $\delta A=Ax+\delta x-A(x)$



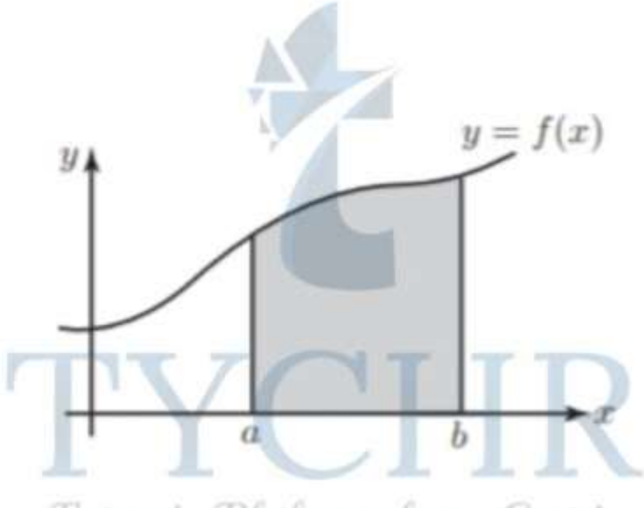
- Now, δA, can be approximated by assuming it has the form of a rectangle of height $y=f(x)$ and width δx.

- That is $\delta A \approx f(x)\delta x$ or $\frac{\delta A}{\delta x} \approx f(x)$
- In the limit as δx tends to zero we have $\frac{dA}{dx} = f(x)$

We can use the previous section to interpret this result.
Given that f(x) is the derivative of A(x) then, A(x) must be an antiderivative of f(x), i.e.
Ax=f(x)+C ①
We wrote f(x)+C for a whole family of functions all of which are antiderivatives when F(x) is. Now, we are looking for a specific antiderivative by choosing a specific value for

1. C. This is obtained by noting that when x=0 the area bounded is zero, that is 0=F0+C, and this gives us a value for C, that is C=-F(0).

We can use this result as follows:
Given a function f(x), then the formula in Equation ① can tell us the area under the graph provided we can calculate an antiderivative F(x).
Suppose now we want to find the area under the graph between x=a and x=b as shown below:



The total area up to x=b is given, using Equation ①, by
A(b)=F(b)+C
The total area up to x=a is given, using Equation ①, by
A(a)=F(a)+C
And so, subtracting, the area between a and b is
A(b)-A(a)=F(b)-F(a)
Note how the C's cancel out.

This area can be denoted by: $A = \int_a^b f(x)dx$
This is what we call **definite integral.**
Properties of definite integral:

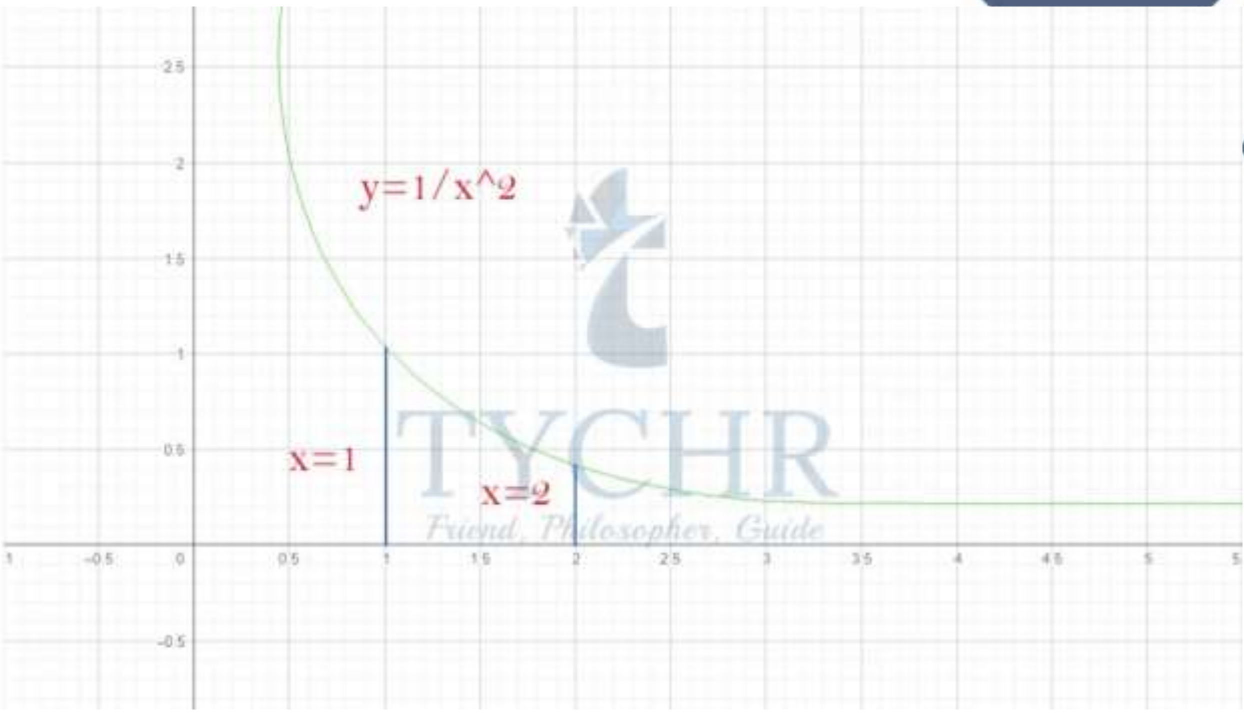
1. Sum or difference: $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
2. Constant multiple: $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
3. Reverse interval: $\int_a^b f(x)dx = -\int_b^a f(x)dx$
4. Zero-length interval: $\int_a^a f(x)dx = 0$
5. Adding intervals: $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

FUNDAMENTAL THEOREM OF CALCULUS

The fundamental theorem of calculus is a theorem that links the concept of differentiating a function with the concept of integrating a function.
The theorem has two versions:

- $\frac{d}{dx}\int_a^x f(t)dt = f(x)$ We start with a continuous function f and we define a new function for the area under the curve y=f(t) $F(x) = \int_a^x f(t)dt$ What this version of the theorem says is that the derivative of F is f. In other words, F is an antiderivative of f. Thus, the theorem relates differential and integral calculus, and tells us how we can find the area under a curve using antidifferentiation.
- $\int_a^b f(x)dx = F(b) - F(a)$ This version gives more direct instructions to finding the area under the curve y=f(x) between x=a and x=b. Simply find an antiderivative F and take F(b)-F(a).

Ex. Calculate the area under the curve $y = \frac{1}{x^2}$ in the domain x=1 to x=2.



$A = \int_{x=1}^{x=2} \frac{1}{x^2}dx$ Now, $F(x) = \int \frac{1}{x^2} = -\frac{1}{x}$ A=F2-F(1)

A=-12-(-1)$A = \frac{1}{2}$ **units**

Ex. Evaluate $\frac{d}{dx}\int_7^x 8t^6 dt$ Using fundamental theorem of calculus: $\frac{d}{dx}\int_7^x 8t^6 dt = 8x^6$

Ex. Evaluate $\int_1^3 \left(\frac{3+u^2}{u^3}\right)du = \int_1^3 \left(\frac{3}{u^3} + \frac{u^2}{u^3}\right)du = \int_1^3 \frac{3}{u^3}du + \int_1^3 \frac{1}{u}du = \left(\frac{3u^{-2}}{-2}\right) + \ln ln\, u$ |2 1

$= \left(-\frac{3}{2(4)^2} + \ln ln\, 2\right) - \left(-\frac{3}{2(1)^2} + \ln ln\, 1\right) = \frac{9}{8} + \ln ln\, 2$

Evaluating definite integrals by substitution:

Ex. Evaluate $\int_2^3 (3x-2)^4 dx$ Putting 3x-2=u

3dx=du

Since we are changing the variable, the limits will also change.

We have 3x-2=u

At x=2, u=4

At x=3, u=7

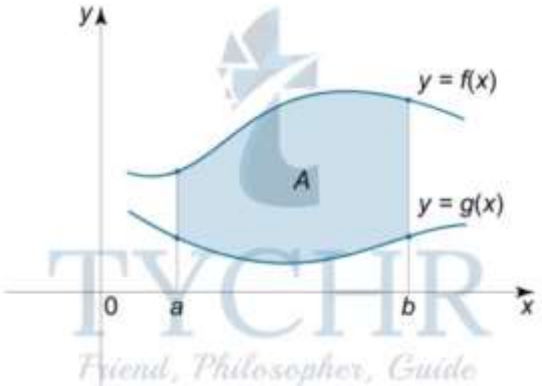
Now, the limits will become from u=4 to u=7 Putting the limits, u and du in the integral: $= \frac{\int_4^7 u^4 du}{3} = \frac{1}{3}\left(\frac{u^5}{5}\right)|u = 7\ u = 4 = \frac{1}{15}(7^5 - 4^5) = 1052.2$

AREA BETWEEN TWO CURVES

Recall that the area under the graph of y=f(x) between x=a and x=b, is given by: $A = \int_a^b f(x)dx = F(b) - F(a)$ Here, F(x) is the antiderivative of f(x)

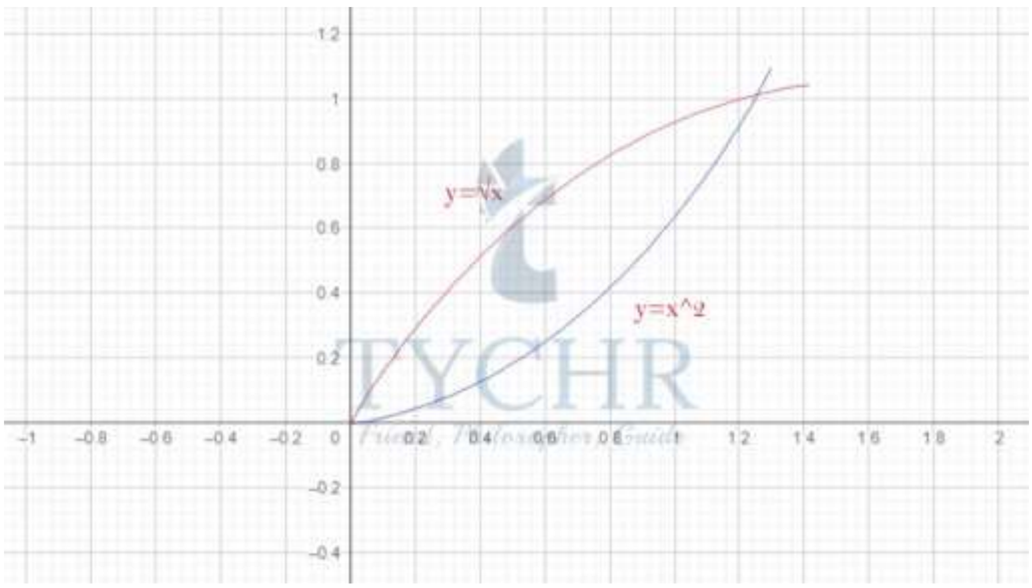
If f(x) and g(x) are two functions, given that f(x)>g(x) in the interval [a,b], then the area between the curves y=f(x) and y=g(x) in this intevral is given by-

$A = \int_a^b [f(x) - g(x)]dx$ In terms of antiderivatives:



$A = \int_a^b [f(x) - g(x)]dx = F(b) - G(b) - F(a) + G(a)$ Here, F(x) and G(x) are antiderivatives of the functions f(x) and g(x) respectively.

Ex. Determine the area of the region enclosed by $y=x^2$ and $y = \sqrt{x}$



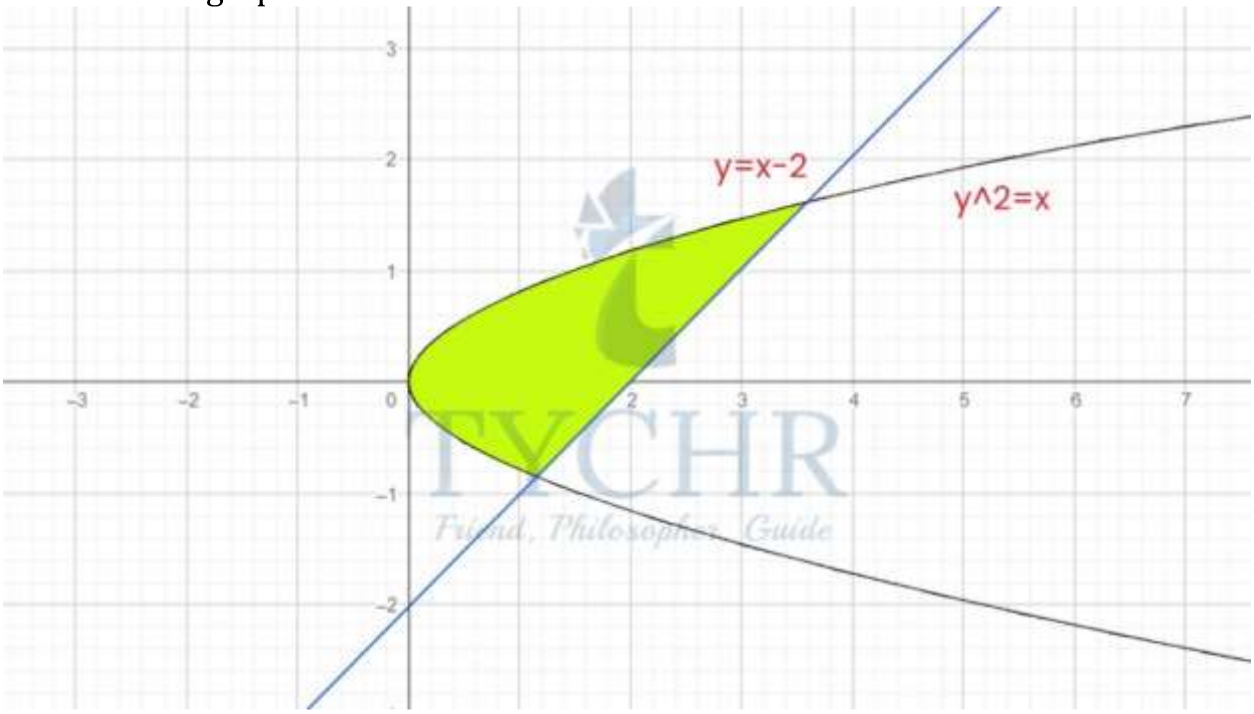
First lets plot the graph of the two functions.
The intersection of the two graphs is at 0,0 and (1,1)

$A = \int_0^1 (\sqrt{x} - x^2)dx$

The limits of the integration will be x=0 and x=1.

$A = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3|1\ 0$ $A = \frac{2}{3}(1) - \frac{1}{3}(1)$ $A = \frac{1}{3}$ **unit**

Ex. Find the area of the region in the first quadrant enclosed by the graphs of y^2=x and y=x-2
Consider the graph:



The graphs here, are along y-axis, so we will write the expressions, in terms of y:
x=y^2
x=y+2

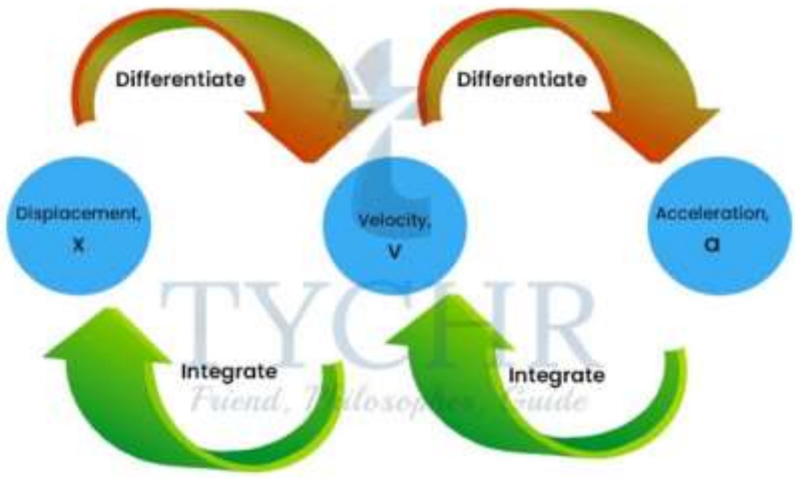
Now, we will determine the intersection of the two curves-
y^2=y+2
(y+1)(y-2)=0
y=-1, y=2

Area = $= A = \int_a^b [f(y) - g(y)]dy$ $A = \int_{-1}^2 (y + 2 - y^2)dy$ $A = \frac{y^2}{2} + 2y - \frac{y^3}{3}|y = 2\ y = -1$

$A = \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$ $A = \frac{9}{2}$ **units**

KINEMATICS USING INTEGRATION

- If v is the velocity function for a particle moving along a straight line, then: $\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1)$ is the displacement from t1 to t2.
- $\int_{t_1}^{t_2} |v(t)|dt$ t gives the total distance travelled from t1 to t2.
- Similarly, the integration of acceleration function will give us the velocity function.



Ex. A particle P is moving along a straight line. The acceleration ams-1 from O can be written as a function of t
a=t+19

Suppose we know that when t=0, we have s=0 and when t=4 we have s=10. Find the displacement when t=10.

First we will find velocity by integrating the acceleration function: $v = \int a dt$

$v = \int (t + 19)dt$ $v = \frac{t^2}{2} + 19t + C_1$

Now, we will integrate velocity to obtain the displacement: $s = \int v dt$

$s = \int \left(\frac{t^2}{2} + 19t + C_1\right)dt$ $s = \frac{t^3}{6} + \frac{19t^2}{2} + C_1 t + C_2$

Now, putting t=0 and s=0
0=0+0+0+C2
C2=0

The displacement now becomes- $s = \frac{t^3}{6} + \frac{19t^2}{2} + C_1 t$

Putting t=4 and s=10 $10 = \frac{64}{6} + \frac{304}{2} + 4C_1 C_1 = -\frac{229}{6}$

The displacement now becomes- $s = \frac{t^3}{6} + \frac{19t^2}{2} - \frac{229}{6} t$
Putting t=10
s=735 **units**

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