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Chapter 10 Integration

Mathematics

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Chapter 10: Integration

Q1) 1 Find the indefinite integral.

(a) $\int x^8 dx$
(b) $\int (5x^4 - 6x^2 + 7)dx$
(c) $\int \sqrt[10]{x^3} dx$
(d) $\int \frac{4}{x^9} dx$
(e) $\int \frac{8x^5+4x}{2x^2} dx$
(f) $\int 4e^x dx$
(g) $\int (6\sqrt{x} + 2)dx$
(h) $\int (x^2 + 3)^2 dx$
(i) $\int (4x + 5)^6 dx$
(j) $\int 6e^{3x+2} dx$
(k) $\int \frac{1}{6x-7} dx$
(l) $\int \ln(e^{3x})\, dx$

Ans 1:

(a) $\int x^8 dx$

$= \frac{x^{8+1}}{8+1} + C$

$= \frac{x^9}{9} + C$

(b) $\int (5x^4 - 6x^2 + 7)dx$

$= \frac{5x^4}{4} - 6\left(\frac{x^3}{3}\right) + 7x + C$

$= \frac{5x^4}{4} - 2x^3 + 7x + C$

(c) $\int \sqrt[10]{x^3} dx$

$= \int x^{3/10} dx$

$= \frac{10}{13} x^{13/10} + C$

(d) $\int \frac{4}{x^9} dx$

$= \int 4\, x^{-9} dx$

$= 4(-8)x^{-8} + C$

$= -\frac{32}{x^8} + C$

(e) $\int \frac{8x^5+4x}{2x^2} dx$

$= \int \frac{8x^5}{2x^2} + \frac{4x}{2x^2} dx$

$= \int 4x^3 + \frac{1}{2}\left(\frac{1}{x}\right) dx$

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$= \frac{4x^4}{4} + \frac{1}{2}\ln x + C$
$= \boldsymbol{x^4 + \ln\sqrt{x} + C}$

(f) $\int 4e^x dx$
$= 4\int e^x dx$
$= \boldsymbol{4e^x + C}$

(g) $\int (6\sqrt{x} + 2)dx$
$= \frac{6x^{\frac{3}{2}}}{3/2} + 2x + C$
$= \boldsymbol{4x^{3/2} + 2x + C}$

(h) $\int (x^2 + 3)^2 dx$
$= \int x^4 + 6x^2 + 9\ \ dx$
$= \frac{x^5}{5} + \frac{6x^3}{3} + 9x + C$
$= \boldsymbol{\frac{x^5}{5} + 2x^3 + 9x + C}$

(i) $\int (4x + 5)^6 dx$
Let $4x + 5 = u$
hence, $4dx = du$
$= \frac{1}{4}\int (4x + 5)^6 4dx$
Putting the value of u and du
$= \frac{1}{4}\int u^6 du$
$= \frac{\frac{1}{4}u^7}{7} + C$
Putting back the value of u
$= \boldsymbol{\frac{1}{28}(4x + 5)^7 + C}$

(j) $\int 6e^{3x+2} dx$
$= 6\int e^{3x+2}$
Let $3x + 2 = u$
$3dx = du$
$= \frac{6}{3}\int e^{3x+2}\, 3dx$
Putting the value of u and du
$= 2\int e^u du$
$= 2e^u + C$
Putting back the value of u
$= \boldsymbol{2e^{3x+2} + C}$

(k) $\int \frac{1}{6x-7} dx$
Let $6x - 7 = u$
$6dx = du$
Putting the value of u and du
$= \frac{1}{6}\int \frac{1}{u} du$
$= \frac{1}{6}\ln u + C$

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Putting back the value of u

$= \frac{1}{6}\ln(6x-7) + C$

(l) $\int \ln(e^{3x})\,dx$

$= \int 3x\,dx$

$= \frac{3x^2}{2} + C$

Q2) Find the definite integral.

(a) $\int_{-2}^{3}(6x-1)dx$

(b) $\int_{-1}^{3} x^2 dx$

(c) $\int_{9}^{25} \frac{3}{\sqrt{x}} dx$

(d) $\int_{1}^{e^4} \frac{5}{x} dx$

(e) $\int_{-2}^{0} 8(2x+3)^3 dx$

(f) $\int_{3}^{5} e^{4x} dx$

Ans 2:

(a) $\int_{-2}^{3}(6x-1)dx$

$= \frac{6x^2}{2} - x\Big|_{x=-2}^{x=3}$

$= 3x^2 - x\Big|_{x=-2}^{x=3}$

$= 3(3)^2 - 3 - (3(-2)^2 - (-2))$

$= 27 - 3 - (12+2)$

$= 10$

(b) $\int_{-1}^{3} x^2 dx$

$= \frac{x^3}{3}\Big|_{x=-1}^{x=3}$

$= \frac{3^3}{3} - \frac{(-1)^3}{3}$

$= \frac{27+1}{3}$

$= \frac{28}{3}$

(c) $\int_{9}^{25} \frac{3}{\sqrt{x}} dx$

$= 3(2\sqrt{x})\Big|_{x=9}^{x=25}$

$= 6(\sqrt{25} - \sqrt{9})$

$= 6(5-3)$

$= 12$

(d) $\int_{1}^{e^4} \frac{5}{x} dx$

$= 5\ln x\Big|_{x=1}^{x=e^4}$

$= 5(\ln e^4 - \ln 1)$

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$= 5(4-0)$
$= \mathbf{20}$

(e) $\int_{-2}^{0} 8(2x+3)^3 dx$
Let $u = 2x+3$
$du = 2dx$
Putting the value of u and du
The limits will also change to -1 to 3
$= \int_{-1}^{3} 4u^3 du$
$= \frac{4u^4}{4}\Big|_{u=-1}^{u=3}$
$= (3^4 - (-1)^4)$
$= 81 - 1$
$= \mathbf{80}$

(f) $\int_{3}^{5} e^{4x} dx$
Let $4x = u$
$4dx = du$
Putting the value of u and du
The limits will also change to 12 to 20
$= \frac{1}{4}\int_{12}^{20} e^u du$
$= \frac{1}{4} e^u\Big|_{u=12}^{u=20}$
$= \mathbf{\frac{1}{4}(e^{20} - e^{12})}$

Q3) Given that $\int_1^4 f(x)dx = 10$ and $\int_1^3 f(x)dx = 6$
Find:
(a) $\int_1^4 2f(x)dx$
(b) $\int_3^4 f(x)dx$
(c) $\int_1^4 (f(x)+4)dx$

Ans 3:
(a) $\int_1^4 2f(x)dx$
$= 2\int_1^4 f(x)dx$
$= 2(10)$
$= \mathbf{20}$

(b) $\int_3^4 f(x)\, dx$
$= \int_1^4 f(x)dx - \int_1^3 f(x)dx$
$= 10 - 6$
$= \mathbf{4}$

(c) $\int_1^4 (f(x)+4)dx$
$\int_1^4 f(x)\, dx + \int_1^4 4dx$
$= 10 + 4x\Big|_{x=1}^{x=4}$
$= 10 + 4(4-1)$

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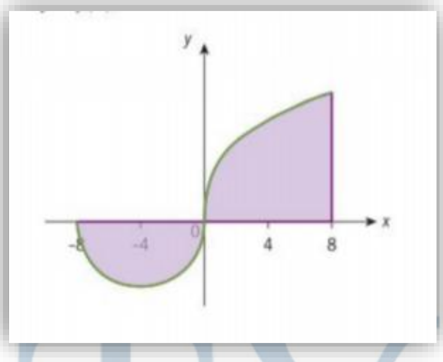
$= 10 + 12$
$= \mathbf{22}$

Q4) Let $f'(x) = 4x^3 + 2$. Given that $f(2) = 24$, Find $f(x)$.
Ans 4: $f'(x) = 4x^3 + 2$
By fundamental theorem of calculus:
$f(x) = \int 4x^3 + 2\,dx$
$f(x) = \frac{4x^4}{4} + 2x + C$
$f(x) = x^4 + 2x + C$

Putting $f(2) = 24$
$24 = 2^4 + 2(2) + C$
$24 = 16 + 4 + C$
$C = 4$

Now, putting the value of C in the function:
$\boldsymbol{f(x) = x^4 + 2x + 4}$

Q5) The following diagram shows the graph of $y = f(x)$, for $-8 \leq x \leq 8$



The first part of the graph is a semicircle with centre$(-4,0)$.
(a) Find $\int_{-8}^{0} f(x)dx$.
(b) The shaded region is enclosed by the graph of f, the x-axis and the line $x = 8$ .The area of this region is 21π. Find $\int_{-8}^{8} f(x)dx$.

Ans 5:
(a) The integral $\int_{-8}^{0} f(x)dx$ will give us the area of the semicircle.
The semicircle has centre at $(-4,0)$ which means the radius is 4 units.
Area $= \frac{\pi r^2}{2} = \frac{\pi 4^2}{2} = 8\pi$
But, the area is below x-axis, therefore the integral will give us the negative value of the area, which gives us:
$\boldsymbol{\int_{-8}^{0} f(x)dx = -8\pi}$

(b) The integral $\int_{-8}^{8} f(x)dx$ will give us the total area shaded in the diagram.
The area of the semicircle is 8π and the area of the other part in the 1st quadrant is 21π.
The semicircle is below x-axis and the other part is above x-axis, therefore:
$\int_{-8}^{8} f(x)dx = -8\pi + 21\pi$

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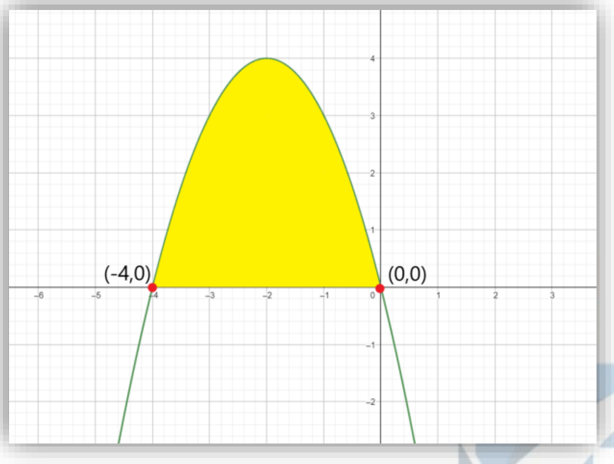
$\int_{-8}^{8} f(x)dx = 13\pi$

Q6) Find the area under the graph of $f(x) = -x^2 - 4x$.
Ans 6: Sketching the graph and shading the required area:



$A = \int_{-4}^{0}(-x^2 - 4x\,)dx$

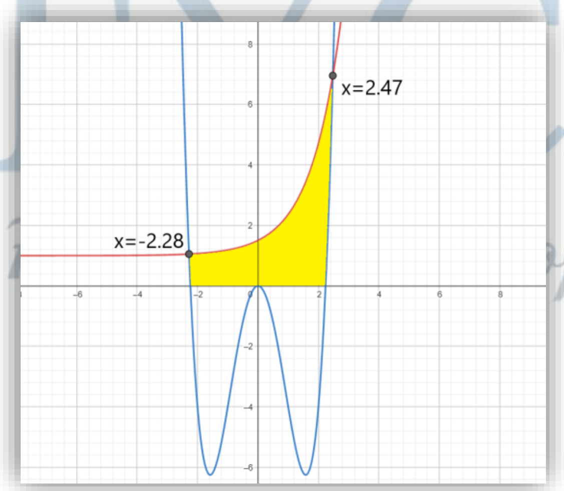
$A = -\frac{x^3}{3} - \frac{4x^2}{2}\Big|_{x=-4}^{x=0}$

$A = 0 - \left(-\frac{(-4^3)}{3} - 2(-4)^2\right)$

$A = 10.66$ **Units**

Q7) Find the area of the region enclosed by the graphs of the given functions.
(a) $f(x) = x^4 - 5x^2$ and $g(x) = 0.5e^x + 1$
(b) $f(x) = x^3 - 9x$ and $g(x) = x^2 - 3x$

Ans 7:
(a) Sketching the graph, shading the required area and marking the intersection points:



$A = \int_{-2.28}^{2.47}(g(x) - f(x))dx$

$A = \int_{-2.28}^{2.47} 0.5e^x + 1 - (x^4 - 5x^2)\,dx$

$A = 0.5e^x + x - \frac{x^5}{5} + \frac{5x^3}{3}\Big|_{x=-2.28}^{x=2.47}$

$A = 0.5e^{2.47} + 2.47 - \frac{(2.47)^5}{5} + \frac{5(2.47)^3}{3} - \left(0.5e^{-2.28} - 2.28 - \frac{(-2.28)^5}{5} + \frac{5(-2.28)^3}{3}\right)$

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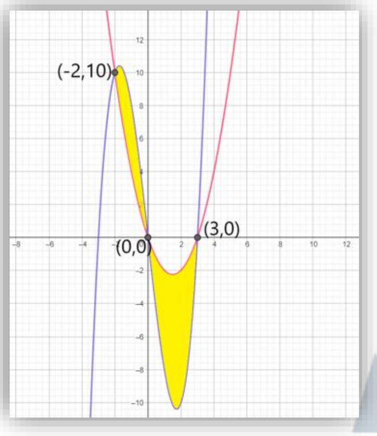
$A = 5.34$ **units**

(b) Sketching the graph, shading the required area and marking the intersection points:



$A_1 = \int_{-2}^{0} f(x) - g(x)dx$

$A_1 = \int_{-2}^{0} x^3 - 9x - (x^2 - 3x)\, dx$

$A_1 = \frac{x^4}{4} - \frac{9x^2}{2} - \frac{x^3}{3} + \frac{3x^2}{2} \Big|_{x=-2}^{x=0}$

$A_1 = -4 + 18 - \frac{8}{3} - 6$

$A_1 = \frac{16}{3}$

$A_2 = \int_{0}^{3} f(x) - g(x)\, dx$

$A_2 = \int_{0}^{3} x^3 - 9x - (x^2 - 3x)\, dx$

$A_2 = \frac{x^4}{4} - \frac{9x^2}{2} - \frac{x^3}{3} + \frac{3x^2}{2} \Big|_{x=0}^{x=3}$

$A_2 = \frac{81}{3} - \frac{81}{2} - 9 + \frac{27}{2}$

$A_2 = -9$

The negative sign indicates that the area is below x-axis.
Total area $= A = A_1 + A_2$

$A = \frac{16}{3} + 9$

$A = 14.33$ **units**

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Q8)

(a) Find the equation of the tangent line to the graph of $f(x) = x^3$ at $x = 1$

(b) The tangent line to the graph of $f(x) = x^3$ at $x = 1$ intersects f at one other point. Find the coordinates of this point.

(c) Find the area of the region enclosed by the graph of f and the tangent line.

Ans 8:

(a) $f(x) = x^3$

$f'(x) = 3x^2$

The slope of the tangent line to $f(x)$ at $x = 1$ will be $m = f'(1)$

$m = 3(1^2)$

$m = 3$

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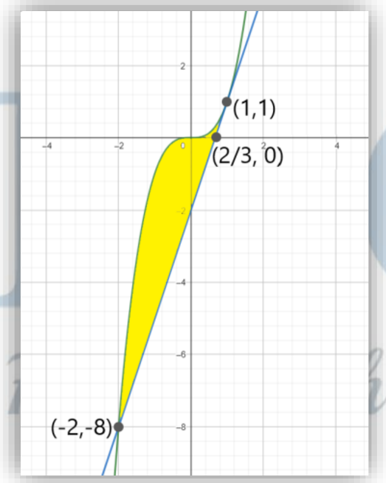
At $x = 1, f(x) = 1$
$m = 3$, point- $(1,1)$
Using point slope form of line
$y - y_0 = m(x - x_0)$
$y - 1 = 3(x - 1)$
$y - 1 = 3x - 3$
$\mathbf{y = 3x - 2}$

(b) We have-
$f(x) = x^3$
Tangent line - $y = 3x - 2$
Solving the above two equations:
$x^3 = 3x - 2$
$x^3 - 3x + 2 = 0$
$(x - 1)(x^2 + x - 2) = 0$
$(x - 1)(x - 1)(x + 2) = 0$
$x = 1, x = -2$

We already know that the tangent line and f intersect at $x = 1$. Now we will take $x = -2$.
Putting $x = -2$ in $f(x)$
$f(-2) = (-2)^3$
$f(-2) = -8$

Required coordinates- $\mathbf{(-2, -8)}$

(c) Sketching the graph and marking the area and intersection point:



$A_1 = \int_{-2}^{0} (3x - 2) - x^3 \, dx$
$A_1 = \frac{3x^2}{2} - 2x - \frac{x^4}{4} \Big|_{x=-2}^{x=0}$
$A_1 = -(6 + 4 - 4)$
$A_1 = -6$
The negative sign indicates that the area is below x-axis

$A_2 = \int_{0}^{2/3} (3x - 2) \, dx$

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$A_2 = \frac{3x^2}{2} - 2x|_{x=0}^{x=2/3}$

$A_2 = \frac{2}{3} - \frac{4}{3}$

$A_2 = -\frac{2}{3}$

The negative sign indicates that the area is below x-axis

$A_3 = \int_0^1 x^3\, dx$

$A_3 = \frac{x^4}{4}|_{x=0}^{x=1}$

$A_3 = \frac{1}{4}$

Total area $= A = A_1 + A_2 + A_3$

$A = 6 + \frac{2}{3} + \frac{1}{4}$

$\boldsymbol{A = 6.916}$ **units**

Q9) A function f is defined by $f(x) = \frac{2x}{x^2+1}$. Find expressions for

(a) $f'(x)$

(b) $\int f(x)dx$

Ans 9:

(a) $f(x) = \frac{2x}{x^2+1}$

Let $p(x) = 2x$

$q(x) = x^2 + 1$

$p'(x) = 2$

$q'(x) = 2x$

Applying quotient rule

$f'(x) = \frac{q(x)p'(x) - p(x)q'(x)}{q(x)^2}$

$f'(x) = \frac{(x^2+1)(2) - (2x)(2x)}{(x^2+1)^2}$

$f'(x) = \frac{2(x^2+1-2x^2)}{(x^2+1)^2}$

$\boldsymbol{f'(x) = \frac{2(1-x^2)}{(1+x^2)^2}}$

(b) $\int f(x)dx$

$= \int \left(\frac{2x}{x^2+1}\right) dx$

Let $u = x^2 + 1$

hence, $du = 2xdx$

Putting the value of u and du in the integral:

$= \int \frac{1}{u} du$

$= \ln u + C$

Putting the value of u

$\boldsymbol{= \ln(x^2 + 1) + C}$

Q10) The velocity, $v\text{ms}^{-1}$ of a particle at time t seconds is given by $v = 40 - 3t$

(a) Find an expression for the acceleration of the particle in ms^{-2}.

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(b) Let s represent the displacement, in metres, of the particle from the origin at a time t.Given that s =10 when $t = 1$, find an expression for s as a function of t.

Ans 10:

(a) Acceleration of the particle $= a = \frac{dv}{dt}$

$a = \frac{d}{dt}(40 - 3t)$

$\boldsymbol{a = -3}$ **ms**$^{-1}$

(b) The displacement of the particle will be $s = \int v\, dt$

$s = \int 40 - 3t\, dt$

$s = 40t - \frac{3t^2}{2} + C$

Now, it is given that $s = 10$ when $t = 1$:

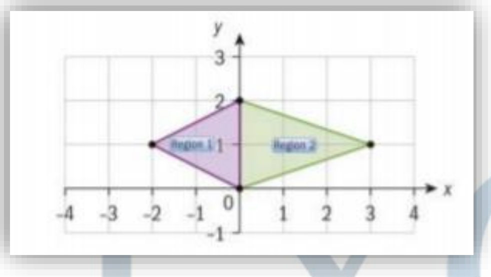
$10 = 40(1) - \frac{3}{2}(1) + C$

$C = -\frac{57}{2}$

The displacement will become-

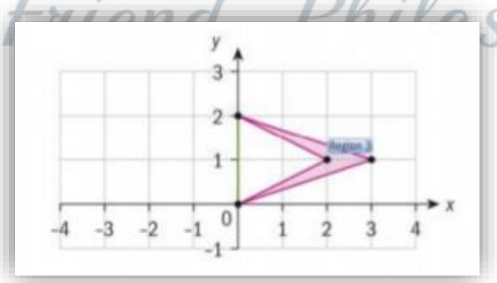
$\boldsymbol{s = 40t - \frac{3t^2}{2} - \frac{57}{2}}$

Q11) Consider the two regions represented on the diagram below. Region 1 is the triangle with vertices (-2,1), (0,0) and (0,2); Region 2 is the triangle with vertices (3,1), (0,0) and (0,2).



(a) Determine the area of :
   (i) Region 1
   (ii) Region 2

(b) Hence determine the area of Region 3, shown on the diagram below



Ans 11:

(a) (i) The area of region 1 will be-

$A_1 = \frac{1}{2} \times b \times h$

The base of the triangle is on the y-axis and is equal to 2.

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The height of the triangle is also 2.

$A_1 = \frac{1}{2}(2)(2)$

$\boldsymbol{A_1 = 2}$ **units**

(ii) The area of region 2 will be-

$A_2 = \frac{1}{2} \times b \times h$

The base of the triangle is on the y-axis and is equal to 2.

The height of the triangle is 3.

$A_2 = \frac{1}{2}(2)(3)$

$\boldsymbol{A_2 = 3}$ **units**

(b) The area of region 3 is formed by taking the reflection of region 1 in y-axis and subtracting it by the area of region 2.

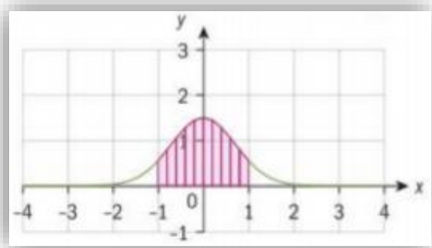
The area of region 3 will be-

$A_3 = A_2 - A_1$

$A_3 = 3 - 2$

$\boldsymbol{A_3 = 1}$ **unit**

Q12) The diagram shows the graph of the function defined by $f(x) = \frac{3e^{-x^2}}{2}$



(a) Find $A = \int_0^1 f(x)\,dx$ correct to 5 significant figures.

(b) Hence find the value of

(i) the area of the shaded region on the diagram.

(ii) $\int_1^2 2f(x-1)\,dx$

Ans 12:

(a) $A = \int_0^1 f(x)\,dx$

$A = \int_0^1 \frac{3e^{-x^2}}{2}\,dx$

The value of integral of e^{-x^2} from zero to 1 is given by $\frac{1}{2}\sqrt{\pi(1 - \frac{1}{e})}$

$A = \frac{3}{2}\left(\frac{1}{2}\right)\sqrt{\pi(1 - \frac{1}{e})}$

$\boldsymbol{A = 1.05690}$ **units**

(b) (i) The given function is symmetric about the y-axis. The area of the shaded region will be twice the area of shaded area on one side of the y-axis. We have evaluated the area of the shaded region on the right of the y-axis in part (a)

Required area $= 2A = 2(1.05690) = \boldsymbol{2.113.}$

(ii) $\int_1^2 2f(x-1)\,dx$

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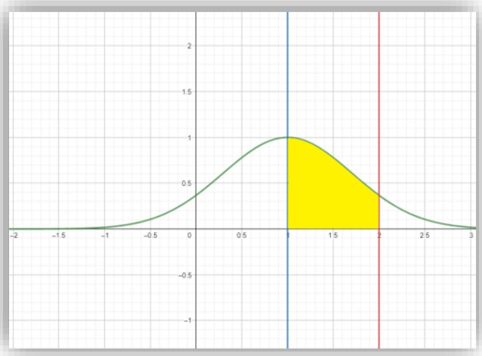
Note the graph of $f(x-1)$:



The area of $f(x-1)$ from $x = 1$ to $x = 2$ is same as the area of $f(x)$ from $x = 0$ to $x = 1$.
Hence,
$\int_1^2 2f(x-1)\,dx = 2(1.05690) = \mathbf{2.113}$

Q13) Consider the function f where $f(x) = x(x^2 - 1)$, $x \in \mathbb{R}$.

(a) Find the coordinates of the points where the graph of f intersects the axes.
(b) Determine an expression for thecderivative of f.
(c) Hence find the x-coordinates of the turning points on the graph of f.
(d) Find the value of $\int_{-1}^{1} f(x)\,dx$.
(e) Explain why the value of the integral found in part d does not represent the area of the region S enclosed by the graph of f and the x-axis between $x = \pm 1$
(f) Find the area of the region S.

Ans 13:

(a) $f(x) = x(x^2 - 1)$
Putting $x = 0$
$f(x) = 0$

Putting $f(x) = 0$
$x(x^2 - 1) = 0$
$x = 0\,, x = 1, x = -1$
The graph intersects the axes at $\mathbf{(0,0), (1,0)}$ **and** $\mathbf{(-1,0)}$

(b) $f(x) = x(x^2 - 1)$
Let $p(x) = x$
$q(x) = x^2 - 1$

$p'(x) = 1$
$q'(x) = 2x$

Applying product rule:
$f'(x) = p(x)q'(x) + q(x)p'(x)$
$f'(x) = x(2x) + (x^2 - 1)(1)$
$\mathbf{f'(x) = 3x^2 - 1}$

(c) f will have turning points at $f'(x) = 0$
$f'(x) = 3x^2 - 1$
$3x^2 - 1 = 0$

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$x^2 = \frac{1}{3}$

$x = \pm\frac{1}{\sqrt{3}}$

f will have turning points at $\boldsymbol{x = \frac{1}{\sqrt{3}}}$ **and** $\boldsymbol{x = -\frac{1}{\sqrt{3}}}$

(d) $\int_{-1}^{1} f(x)\,dx$

$= \int_{-1}^{1} x(x^2 - 1)\,dx$

$= \int_{-1}^{1} x^3 - x\,dx$

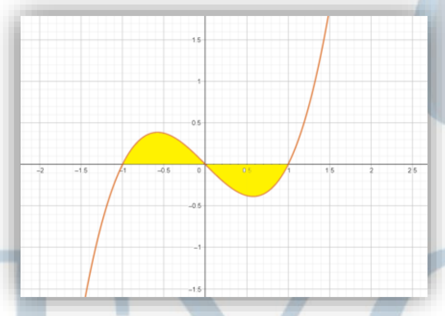
$= \frac{x^4}{4} - \frac{x^2}{2}\Big|_{x=-1}^{x=1}$

$= \frac{1}{4} - \frac{1}{2} - \left(\frac{1}{4} - \frac{1}{2}\right)$

$\boldsymbol{= 0}$

(e) The area enclosed by the region S has some part above and some part below x-axis. If we will integrate directly from -1 to 1, the part above the x-axis will be added and the part below the x-axis will be subtracted, which will not lead us to the total area enclosed by region S. Since the resut we obtained in part (d) is negative, in this case, both the areas (below and above x-axis) must be equal.

(f) We will find the area of region S in two parts. First, consider the graph of $f(x)$:



$A_1 = \int_{-1}^{0} f(x)\,dx$

$A_1 = \int_{-1}^{0} x^3 - x\,dx$

$A_1 = \frac{x^4}{4} - \frac{x^2}{2}\Big|_{x=-1}^{x=0}$

$A_1 = 0 - \left(\frac{1}{4} - \frac{1}{2}\right)$

$A_1 = \frac{1}{4}$

$A_2 = \int_{0}^{1} f(x)\,dx$

$A_2 = \int_{0}^{1} x^3 - x\,dx$

$A_2 = \frac{x^4}{4} - \frac{x^2}{2}\Big|_{x=0}^{x=1}$

$A_2 = \left(\frac{1}{4} - \frac{1}{2}\right) - 0$

$A_2 = -\frac{1}{4}$

The negative sign indicates that the area is below the x-axis.

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Total area $= A_1 + A_2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ **units**

Q14) Consider the functions f and g defined by $f(x) = x^2$ and $g(x) = 3 - 2x$.

(a) Show that the graphs of f and g intersect at points with x- coordinates -3 and 1.

(b) Hence find the area enclosed by the graphs of f and g.

Ans 14:

(a) $f(x) = x^2$ and $g(x) = 3 - 2x$

Finding intersection points:

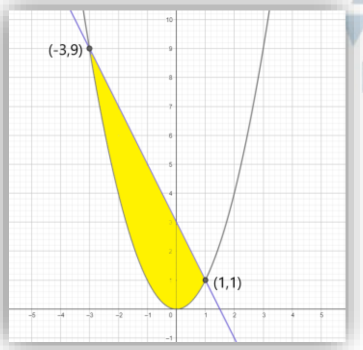
$x^2 = 3 - 2x$

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$\boldsymbol{x = -3\,,\, x = 1}$

(b) Sketching the graph of f and g and marking the area enclosed:



$A = \int_{-3}^{1}(g(x) - f(x))dx$

$A = \int_{-3}^{1} 3 - 2x - x^2\, dx$

$A = 3x - \frac{2x^2}{2} - \frac{x^3}{3}\Big|_{x=-3}^{x=1}$

$A = 3 - 1 - \frac{1}{3} - (-9 - 9 - 9)$

$\boldsymbol{A = 28.67}$ **units**

Q15)

(a) Expand $(x-2)^4$

(b) Hence find $\int (x-2)^4\, dx$

Ans 15:

(a) Using binomial expansion

$(x-2)^4 = {}^4_0Cx^4 + {}^4_1Cx^3(-2) + {}^4_2Cx^2(-2)^2 + {}^4_3Cx(-2)^3 + {}^4_4C(-2)^4$

$\boldsymbol{(x-2)^4 = x^4 - 8x^3 + 24x^2 - 32x + 16}$

(b) $\int (x-2)^4\, dx$

$= \int (x^4 - 8x^3 + 24x^2 - 32x + 16)\, dx$

$= \frac{x^5}{5} - \frac{8x^4}{4} + \frac{24x^3}{3} - \frac{32x^2}{2} + 16x + C$

$\boldsymbol{= \frac{x^5}{5} - 2x^4 + 8x^3 - 16x^2 + 16x + C}$

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Q16) Use the fundamental theorem of calculus to show that $\int_{-1}^{2} |x|\, dx = 2.5$

Ans 16: The fundamental theorem of calculus says –

$\int_a^b f(x)dx = F(b) - F(a)$

Here F is the integral of f.

We have a modulus function in the integral, so we will break the integral first:

$I = \int_{-1}^{0} -xdx + \int_0^2 x\, dx$

$I = -\int_{-1}^{0} x\, dx + \int_0^2 x\, dx$

$I = -\big(F(0) - F(-1)\big) + (F(2) - F(0))$

Now, $\int x\, dx = \frac{x^2}{2} + C$

Computing $F(-1), F(0)$ and $F(2)$:

$F(-1) = \frac{1}{2} + C$

$F(0) = 0 + C$

$F(2) = 2 + C$

$I = -\left(C - \frac{1}{2} - C\right) + (2 + C - 0 - C)$

$I = \frac{1}{2} + 2$

$I = \frac{5}{2}$

$I = \mathbf{2.5}$

Q17) Consider the function f defined by $f(x) = 3xe^{-x}$ on the interval $I = [0,5]$.

(a) Sketch the graph of f showing clearly the coordinates of the maximum point M and end-points A and B.

(b) State the range of f.

(c) Find the equation of the line AB.

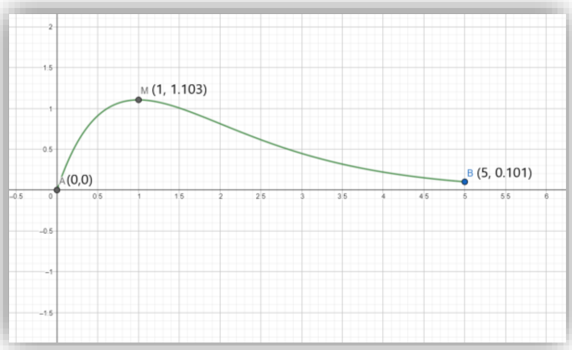
(d) Show that $f'(x) = (3 - 3x)e^{-x}$.

(e) Hence find the equation of the tangent to the graph of f that is parallel to AB. Give all the coefficients in your equation correct to 3 significant figures.

(f) Find the area of the region enclosed by the line AB and the graph of f.

Ans 17:

(a) Graph:



(b) The range of f clearly from the graph is $y \in [\mathbf{0}, \mathbf{1.103}]$ (on the interval [0,5])

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(c) The coordinates are $A(0,0)$ and $B(5, 0.101)$

The slope will be $m = \frac{0.101-0}{5-0} = \frac{1.101}{5}$

Using point slope form of line:

$y - y_0 = m(x - x_0)$

$y - 0 = \frac{1.101}{5}(x - 0)$

$5y = 0.101x$

$\boldsymbol{y = 0.0202x}$

(d) $f(x) = 3xe^{-x}$

$p(x) = 3x$

$q(x) = e^{-x}$

$p'(x) = 3$

$q'(x) = -e^{-x}$

Using product rule:

$f'(x) = p(x)q'(x) + q(x)p'(x)$

$f'(x) = 3x(-e^{-x}) + 3e^{-x}$

$f'(x) = 3e^{-x}(1 - x)$

$\boldsymbol{f'(x) = (3 - 3x)e^{-x}}$

(e) The slope of the tangent will be equal to the slope of AB

$m = \frac{0.101}{5}$

The slope at a point x on $f(x)$ is equal to $f'(x)$

$f'(x) = \frac{0.101}{5}$

$(3 - 3x)e^{-x} = \frac{0.101}{5}$

$x = 0.982$

The value of $f(x)$ at $x = 0.982$:

$f(x) = 3(0.982)e^{-0.982}$

$f(x) = 1.103$

The point of tangency is $(0.982, 1.103)$

Using point slope form of line:

$y - y_0 = m(x - x_0)$

$y - 1.103 = 0.0202(x - 0.982)$

$\boldsymbol{y = 0.020x + 1.08}$

(f) $A = \int_0^5 f(x) - (0.020x)dx$

$A = \int_0^5 (3xe^{-x} - 0.020x)dx$

First we need the value of the integral $\int 3x\, e^{-x} dx$

$\int 3x\, e^{-x} dx = -3e^{-x}(x + 1) + C$

$A = -3e^{-x}(x + 1) - \frac{0.02x^2}{2} \Big|_{x=0}^{x=5}$

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$A = -3e^{-5}(6) - 0.2525 - (-3)$

$\boldsymbol{A = 2.626}$ **units**

Q18) Consider the function defined by $f(x) = 4x - x^3$ for $-2 \leq x \leq 2$.

(a) Complete the table:

x	$\frac{1}{2}$	1	$\frac{3}{2}$
$f(x)$			

(b) Sketch the graph of f, showing clearly the axes intercepts.

(c) Use four rectangles with equal widths to find an approximation of $\int_0^2 f(x)\, dx$.

(d) Find the exact value of $\int_0^2 f(x)\, dx$.

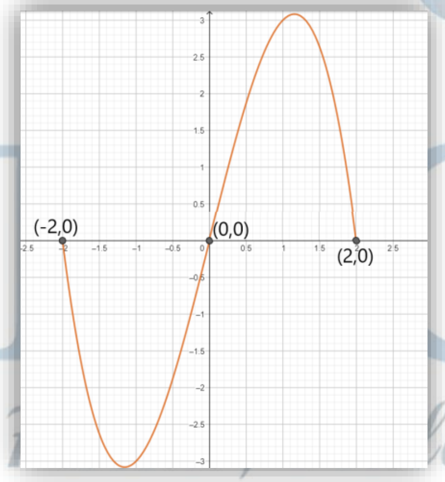
(e) State with reason, the value of $\int_{-2}^{2} |f(x)|\, dx$.

Ans 18:

(a) Completed table:

x	$\frac{1}{2}$	1	$\frac{3}{2}$
$f(x)$	15/8	3	21/8

(b) Graph:



(c) We will consider the four rectangles with width 0.5 as marked in the graph below:

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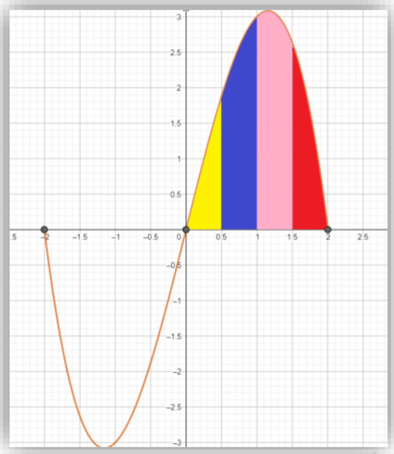
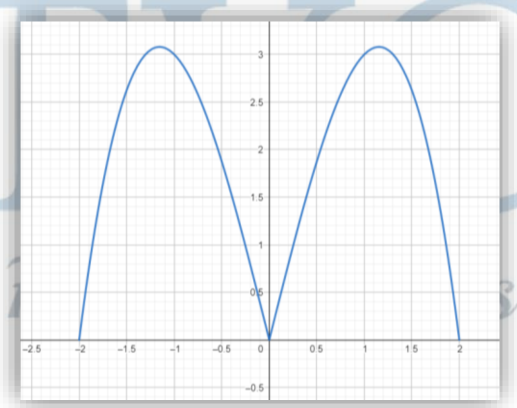


$\int_0^2 f(x)\,dx$ represents the under of these 4 rectangles.
Taking approximate lengths of the rectangles:
$\int_0^2 f(x)\,dx = 0.5(1.5 + 2.5 + 3 + 1.5)$
$\int_0^2 f(x)\,dx = \mathbf{4.25}$

(d) $\int_0^2 f(x)\,dx$
$= \int_0^2 (4x - x^3)\,dx$
$= 2x^2 - \frac{x^4}{4}\Big|_{x=0}^{x=2}$
$\mathbf{= 4}$

(e) Consider the graph of $y = |f(x)|$:



The value of the integral $\int_{-2}^2 |f(x)|\,dx$ will be the area between the above curve and the x-axis from -2 to 2.
Due to symmetry of the graph, the area between the graph and x-axis from -2 to 0 and from 0 to 2 will be equal.
$\int_{-2}^2 |f(x)|dx = \int_{-2}^0 |f(x)|dx + \int_0^2 |f(x)|dx$
$\int_{-2}^2 |f(x)|\,dx = 2\int_0^2 |f(x)|dx$
$\int_{-2}^2 |f(x)|\,dx = 2(4) = \mathbf{8}$

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