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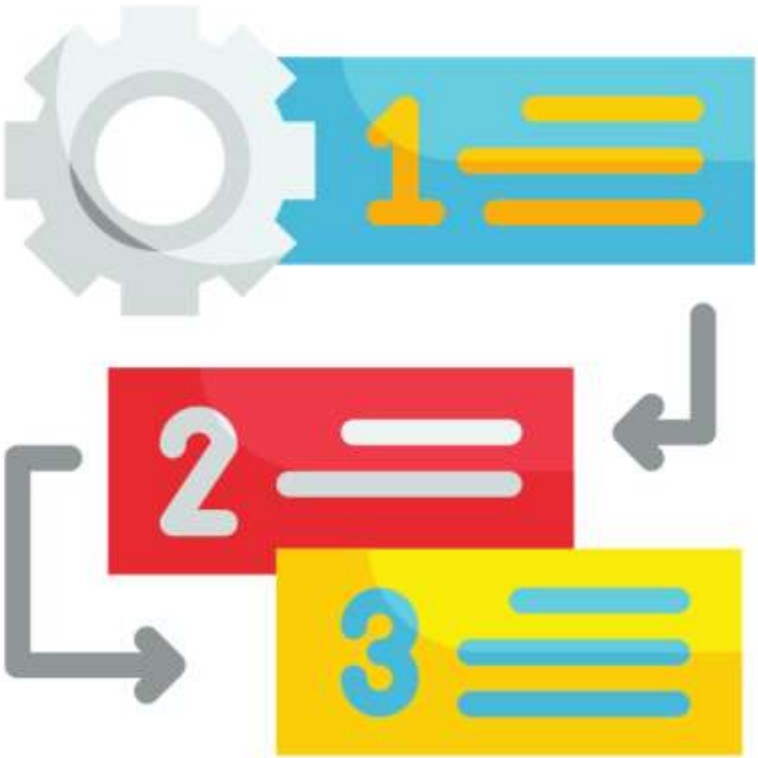
Friend, Philosopher, Guide

Sequences and Series



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SEQUENCES AND SERIES

By the end of this chapter you should be familiar with:

- n^{th} term for Arithmetic and Geometric sequences
- Sum of n terms for Arithmetic and Geometric series
- Using Sigma Notation
- Applications
- Annuities and Amortization

A **sequence** is an arrangement of any objects or a set of numbers in a particular order followed by some rule. If a_1, a_2, a_3, a_4,……… etc., denote the terms of a sequence, then 1,2,3,4,…..denotes the position of the term.
A **sequence** can be defined based upon the number of terms i.e. either finite sequence or infinite sequence.
If a_1, a_2, a_3, a_4, ……. is a sequence, then the corresponding **series** is given by
$S_N = a_1+a_2+a_3 + .. + a_n$
Series a number of events, objects, or people of a similar or related kind coming one after another.
Note: The series is finite or infinite depending if the sequence is finite or infinite.

SEQUENCE	SERIES
Set of elements that follow a pattern	Sum of elements of the sequence
Order of elements is important	Order of elements is not so important
Finite sequence: 1,2,3,4,5	Finite series: 1+2+3+4+5
Infinite sequence: 1,2,3,4,……	Infinite Series: 1+2+3+4+……

TYPES OF SEQUENCE AND SERIES

- Arithmetic
- Geometric
- Harmonic
- Fibonacci Numbers

Where we'll be studying arithmetic and geometric in detail.

Harmonic Sequences

A series of numbers is said to be in harmonic sequence if the reciprocals of all the elements of the sequence form an arithmetic sequence.

Fibonacci Numbers

Fibonacci numbers form an interesting sequence of numbers in which each element is obtained by adding two preceding elements and the sequence starts with 0 and 1. Sequence is defined as, $F_0 = 0$ and $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$

ARITHMETIC SEQUENCES

A sequence a1, a2, a3, …., an is an arithmetic sequence if there is a constant d for which

$\mathbf{a_n = a_{n-1} + d}$

for all integers n > 1, d is called the common difference of the sequence, and d $= a_n - a_{n-1}$ for all integers n > 1.

The general (nth) term of an arithmetic sequence, a_n, with first term a_1 and common difference d, may be expressed explicitly as:

$\mathbf{a_n = a_1 + (n - 1)d}$

Example 1: f a sequence has a first term of $a_1 = 12$ and a common difference d = −7. Write the formula that describes this sequence. Use the formula of the arithmetic sequence.

Solution: As we know, $a_n = a_1 + (n-1)d$

$a_n = 12 + (n-1)(-7) = 12 - 7n + 7$

$a_n = -7n+19$

Example 2: Write down the first five terms of the arithmetic progression with first term 8 and common difference 7.

Solution: We have a=8 and d=7.

The form of an **arithmetic progression** is a, a+d, a+2d, a+3d, a+4d so using these values of a and d the first five terms are:

= 8, 8+7, 8+(2×7), 8+(3×7), 8+(4×7)

= 8, 15, 22, 29, 36

Simple Interest

Suppose you deposit an amount of a into a bank. We refer to **a** as **the principal balance**. You are paid 15% interest on your deposit at the end of each year (per annum). At the end of the first year you will have a total of:

$$(a + 0.15 \times a) = a(1+0.15) = 1.15a$$

With simple interest, the key assumption is that you withdraw the interest from the bank as soon as it is paid and deposit it into a separate bank account. This means that the interest paid each year is only ever paid on the principal balance. At the end of the second year you will therefore have a total of:

$$(a+0.15a+0.15a) = (a(1+2 \times 0.15))$$

At the end of the third year you will have:
$(a+ 0.15a +0.15a+ 0.15a) = (a(1+3 \times 0.15))$
and so on till n years i.e., $(a(1+n \times 0.15))$

We can see that the **total balance**-the principal balance and the interest earned each year up to the present year-forms an arithmetic progression with **first term** a and common difference 0.15a.
In general, for an **interest rate of r per annum** and principal balance of A, the total balance will form an arithmetic progression with first term A and **common difference d=r × a.** Using the above formula for the nth term of an arithmetic progression, **the total balance at the *end* of n years is** a + nd.
Example: James deposits £2,000 into a bank which pays an annual interest rate of 4%. He withdraws and spends the interest earned every time it is paid. Calculate:

a. James' total balance after 55 years
b. The interest that he receives after 55 years

Solution: a) As James withdraws the interest earned each year, his total balance forms an arithmetic sequence with first term a=2,000 and common difference d = 0.04×2000 = 80. We want to calculate his total balance after 5 years. Using the formula with n=5 we have:
Total balance = £ (2,000 + (5×80)) = £ 2,400
b) The interest that James receives over 5years is equal to the difference between his total balance after 5years and his principal sum. We know that James' principal sum is £2,000 and we found in part a) that his total balance

after 5 years is £2,400. The interest received over 5 years is thus:
£ 2,400 – £ 2,000 = £ 400

GEOMETRIC SEQUENCES

A sequence a_1, a_2, a_3, …, a_n is a geometric sequence if there is a constant r for which
$\mathbf{a_n = a_{n-1} \times r}$ for all integers n > 1
r is called the common ratio of the sequence, and r = an ÷ an-1 for all integers n > 1. The general (nth) term of a geometric sequence, an, with common ratio r and first term a1, may be expressed explicitly as
$\mathbf{a_n = a_1 \times r^{n-1}}$

Example 1: Write down the first five terms of the geometric progression which has first term 1 and common ratio 12.

Solution: We have a = 1 and r = 1/2. The general form of a geometric progression is a, ar, ar^2, ar^3, ar^4,… so using these values of a and r the first five terms are:
$1, 1\times(1/2), 1\times(1/2)^2, 1\times(1/2)^3, 1\times(1/2)^4$
1, 1/2 , 1/4, 1/8, 1/16
As we can see, the terms are getting smaller and smaller.

Example 2: Find the 10th term of the geometric progression with first term 3 and common ratio 2.

Solution: We have a=3, r=2 and n=10. Using the formula for the nth term, the 10th term is
$3 \times 2^{10-1} = 3 \times 2^9 = 1536$

Compound Interest

The difference between simple interest and compound interest is that with **compound interest** it is assumed that you do *not* withdraw the interest earned from the bank each time it is paid. At the *end* of each year, the interest will therefore be paid on the total balance earned so far (the principal sum and interest earned each year up to the present year).
If the **principal sum** is **a** and the **interest rate** is **r per annum** the total balance at the *end* of the first year will be: a(1+r)

At the *end* of the second year it will be: $a(1+r)^2$
At the *end* of the third year it will be: $a(1+r)^3$
and so on.
We can see that the total balance forms a geometric progression with first term a and common ratio (1+r). Using the above formula for the nth term of a geometric progression with n−1 replaced by n, the total balance at the *end* of n years is: $\mathbf{a(1+r)^n}$

Note: The total balance at the *beginning* of the nth year is given by: $\mathbf{a(1+r)^{n-1}}$
Now suppose that the bank pays you interest on your deposit t times per year. For example, interest might be paid monthly, in which case we would have t=12. Assuming that the interest earned is not withdrawn after each payment, the total balance after n years is given by: $\mathbf{a(1 +(r/t))^{tn}}$
where, as before, r is the interest rate and A is the principal sum.

Example: After discovering that his wife is in fact having triplets, James takes up a part time job to pay for their combined clothing costs and decides that he will enough money into a bank account to pay for a car worth 3,000 for each of them on their 18th birthday. Assuming that the interest rate is 25% and that James does not withdraw the interest payment each year, how much does James now need to deposit into the bank account?

Solution: As James leaves the interested earned each year in the account, his total balance forms a geometric sequence with first term a and common ratio 1+0.25 = 1.04. Since each car costs £3,000, the combined cost is 3×3000=9000. This is the total balance that James must have after 18 years. Using the formula with n=18, we have:
$9{,}000 = a(1.25^{18}) = 55.511a$
Rearranging and solving this equation for a gives:

a = 162.129

So, James needs to deposit £162.129 in order to have a total balance of at least £9,000 by his children's 18th birthday.

SIGMA NOTATION

The **Summation Operator ∑** is used to denote the sum of a sequence. We call the sum of the terms in a sequence a series. The starting index is written underneath and the final index above, and the sequence to be summed is written on the right. A series does not have to be the sum of all the terms in a sequence.

Rules:

Constant Rule $\sum_{i=1}^{n} a = na$

Constant Multiple Rule $\sum_{i=1}^{n} ax_i = a \sum_{i=1}^{n} x_i$

The Sum of Sequences Rule $\sum_{i=1}^{n} x_i + y_i = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$

ARITHMETIC SERIES

Arithmetic series is the sum of the terms of an Arithmetic sequence $a_1, a_2, a_3, \ldots, a_n$. The sum, S_n, of n terms of an arithmetic series with common difference d, first term a_1, and nth term a_n is
$S_n = (n(a_1 + a_n)) / 2$ or **$S_n = (n(2a_1 + (n - 1)d) / 2$**

Arithmetic mean is given by: **$a_{n-1} = (a_n + a_{n-2})/2$**
This means in the sequence above $a_2, a_3, a_4, \ldots, a_{n-1}$ are arithmetic mean between a_1 and a_n

Example: Find the sum of the first 40 terms of the arithmetic series 2+5+8+11+...
Solution: First find the 40th term:
$a_{40} = a_1 + (n-1)d = 2 + 39(3) = 119$
Then find the sum: $S_n = n(a_1 + a_n)/2$ $S_{10} = 40(2 + 119)/2 = 2420$

GEOMETRIC SERIES

Geometric series is the sum of the terms of a geometric sequence $a_1, a_2, a_3, \ldots, a_n$
The sum, S_n, of a finite geometric series with first term a, and common ratio r is given by:
$S_n = a((1-r^n)/(1-r))$
The sum, S, of an **infinite geometric series** with first term a, such that the common ratio r satisfies the condition $|r| < 1$ is given by:
$S = a/(1-r)$

Geometric mean is the central number in a geometric progression given by:
$a_2 = \sqrt{}$([illegible] × [illegible])

Example 1: Find the sum of the series $S_7 = 1 - 1/\sqrt{2} + 1/2 - 1/2\sqrt{2} + 1/4 - 1/4\sqrt{2} + 1/8$

Solution: This is a geometric progression with $r = -1/\sqrt{2}$. Since the sum of a

$$S_n = \frac{a(1-r^n)}{1-r}$$

geometric progression is given by

$$S_7 = \frac{1(1-(1/\sqrt{2})^7)}{1-(\frac{1}{\sqrt{2}})} = \frac{8\sqrt{2}-1}{8(\sqrt{2}+1)}$$

we have,

Example 2: Solve the equation $x^2 - 2x^3 + 4x^4 - 8x^5 + \ldots = 2x+1$, $|x|<1$

Solution: We know that, S =a/(a-r)

$x^2 - 2x^3 + 4x^4 - 8x^5 + \ldots = x^2(1 - 2x + 4x^2 - 8x^3 + \ldots) = x^2/(1-(-2x)) = x^2/(1+2x)$

Then, $x^2/(1+2x) = 2x + 1$

$3x^2 + 4x + 1 = 0 = -1$ or $-1/3$ As $|x|<1$, $x = -1/3$

	Arithmetic Progression	Geometric Progression
Sequence	a, a+d, a+2d,……,a+(n-1)d,….	a, ar, ar^2,….,$ar^{(n-1)}$,…
Common Ratio	Successive term – Preceding term Common difference = d = $a_2 - a_1$	Successive term/Preceding term Common ratio = r = $ar^{(n-1)}/ar^{(n-2)}$
General Term (nth Term)	$a_n = a + (n-1)d$	$a_n = ar^{(n-1)}$
nth term from the last term	$a_n = l - (n-1)d$	$a_n = 1/r^{(n-1)}$
Sum of first n terms	$s_n = n/2(2a + (n-1)d)$	$s_n = a(r^n)/(1-r)$ if r 1 $s_n = a(r^n - 1)/(r - 1)$………(2) if r > 1

ANNUITIES AND AMORTIZATION

Annuities are a sequence of equal payments made at regular time intervals and **amortization** determines sequence of payments.

Present Value of Annuity:

$$PV = P \times \frac{1-(1+r)^{-n}}{r} \text{ or } PV = P \times \frac{(1+r/n)^{nt} - 1}{r/n}$$

$$P = PV \times \frac{r}{1-(1+r)^{-n}} \text{ or } P = PV \times \frac{r/n}{(1+r/n)^{nt} - 1}$$

Amortization:

Example 1: Jolene deposits $100 every month into an annuity at 5% compounded monthly, for 9 years. Find the future value and the interest her account earned.

Solution: By using the formula $PV = P \times \frac{(1+r/n)^{nt} - 1}{r/n}$

Jolene's future value will be

$$= 100 \times \frac{(1+0.05/12)^{12x9} - 1}{0.05/12} = \$13{,}604.32$$

Example 2: Dave wants to save $20,000 for a down payment on a really cool speedboat. He opens an annuity at 4.25% compounded quarterly for 3 years. What is his quarterly payment?

Solution: By using the formula $PV = P \times \frac{(1+r/n)^{nt} - 1}{r/n}$

$$20000 = P \times \frac{(1+0.0425/4)^{4x3} - 1}{0.0425/4}$$

$$P = \frac{20000}{12.726} = \$1571.50$$

$1571.50 is the quarterly deposit.

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