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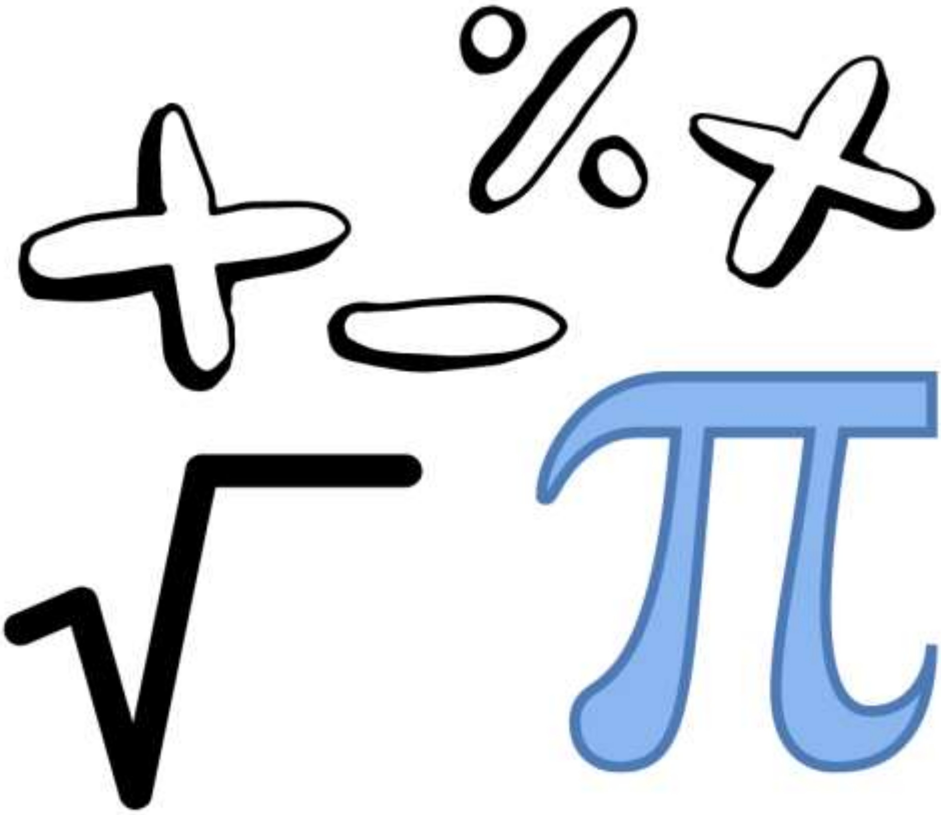
Friend, Philosopher, Guide

Complex Number



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By the end of this chapter you should be familiar with:

- The properties of complex numbers
- Cartesian form of complex numbers
- Polar form of complex numbers
- Exponential form of complex numbers
- Powers and roots of complex numbers
- The use of complex numbers in STEM applications

Real numbers are simply the combination of rational and irrational numbers, in the number system. **Complex numbers** cannot be represented on the number line, but are analytic solutions to equations whose solutions are not real numbers. These complex numbers have an **imaginary unit**.

IMAGINARY NUMBERS

A number that is expressed in terms of the square root of a negative number is called an **imaginary number**. The imaginary unit is i and **$i = \sqrt{(-1)}$**

PROPERTIES:

- $i = \sqrt{(-1)}$
- $i^2 = -1$
- $i^3 = -i$
- $i^4 = 1$, then $i^5 = i$ and so on for i^n

Every power n is an integer multiple of four that produces $i^n = 1$ and the pattern repeats.

Example: Express each in its simplest form

a) $i^{32} = i^4 = 1$

b) $i^{126} = i^2 = -1$

Complex number has two parts, a **real part** and an **imaginary part** which can be thought of as two- dimensional numbers. This concept can be understood by placing them in the complex plane called the **argand plane.** A complex number has the form **$z = a + bi$**.

|z| is the **modulus** of the complex number and the **argument of z** is defined as the **angle between the positive real axis and the line from origin to z.**

Example: Find the zeros of the equation $y = x^2 - 2x + 5$ and find the modulus and argument of the zeros.

Solution:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm \sqrt{-16}}{2}$$

$$|x| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$x = \frac{2 \pm 4i}{2} = 1 \pm 2i$$ Arg(x) = $\tan^{-1}(2/1) = 63.435^{\circ}$

When a quadratic polynomial in x has zeros r_1 and r_2 then $(x - r_1)$ and $(x - r_2)$ are its factors. Then the quadratic equation is $= x^2 - (r_1 + r_2)x + r_1r_2$ Therefore, we can conclude that **the negative sum of the zeroes is the coefficient of x** and the **product of the zeros is the constant term.**

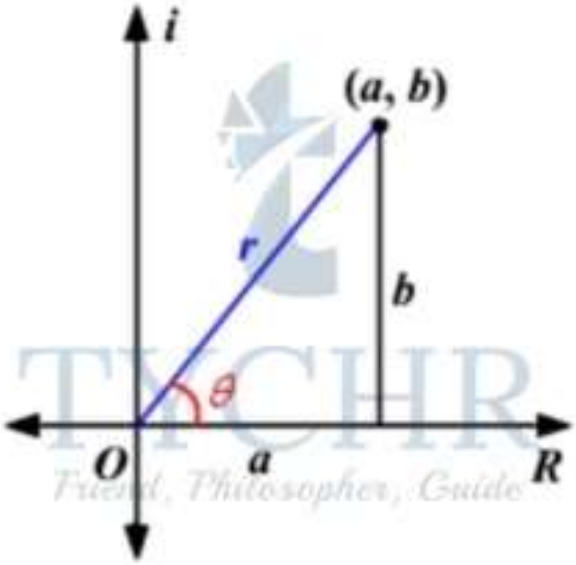
The complex conjugate of $z = a + bi$ is $z^* = a - bi$ and the product of zz^* is $(a + bi)(a - bi) = a^2 + b^2$

Example: Rationalize in the form a + bi.
Solution: $6/(1+i) = (6/(1+i)) \times ((1-i)/(1-i)) = 6(1-i)/2$
$= 3(1 - i) = 3 - 3i$

POLAR FORM OF COMPLEX NUMBERS

The **polar form** of a complex number z = a + bi is **z = r(cosθ + isinθ)** and can be abbreviated as z = r cis

where **$r=|z|=\sqrt{(a^2 + b^2)}$** (modulus) **$a = r\cos\theta$** and **$b = r\sin\theta$**
$\theta = \tan^{-1}(b/a)$ (argument also referred to as arg(z)) **for** $a > 0$ and = $\tan^{-1}(b/a) + \pi$ or $\theta = \tan^{-1}(b/a) +$ **$180°$ for** $a < 0$.



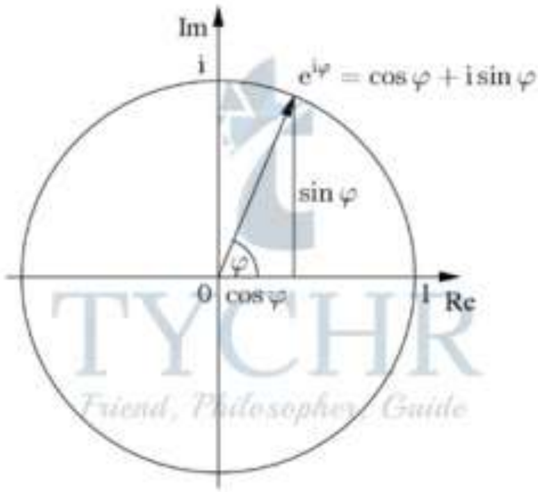
Example: Express 5 + 2i complex number in polar form.
Solution: The polar form of a complex number $z = a + bi$ is $z=r(\cos\theta + i\sin\theta)$.
So, first find the absolute value of r.
$r=|z|= \sqrt{(a^2 + b^2)} = \sqrt{(5^2 + 2^2)} = \sqrt{29} = 5.39$

Now find the argument θ.
Since $a > 0$, use the formula $\theta=\tan^{-1}(b/a)$.
$\theta = \tan^{-1}(2/5) = 0.38$

Note that here θ is measured in radians.
Therefore, the polar form of 5 + 2i is about $5.39(\cos(0.38) + i\sin(0.38))$.

EULER FORM OF COMPLEX NUMBERS

Euler's formula is the statement that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. When x = π, we get Euler's identity, $e^{i\pi} = -1$, or $e^{i\pi} + 1 = 0$. Basically, $\mathbf{z = a + bi =}$



$\mathbf{re^{i\theta}}$

Example: Taking the last example where we found r = 5.39 and = 0.38.5 + 2i can also be expressed in Euler's form as $z = 5.39\ e^{i0.38}$

POWERS OF COMPLEX NUMBERS

DeMoivre's Theorem states that $z^n = (re^{i\theta})^n = r^n e^{i\theta}$

or $\mathbf{z^n = r^n\ cis(n\theta)}$

Example: Compute $(3 + 3i)^5$

$r=|z|= \sqrt{(a^2 + b^2)} = \sqrt{(3^2 + 3^2)} = 3\sqrt{2} = \tan^{-1}(3/3) = \pi/4$

$z^n = (re^{i\theta})^n$

$(3 + 3i)^5 = (3\sqrt{2})^5\ e^{i\,5\pi/4} = 972\sqrt{2}(\cos(5\pi/4) + i\sin(5\pi/4)) = -972 - 972i$

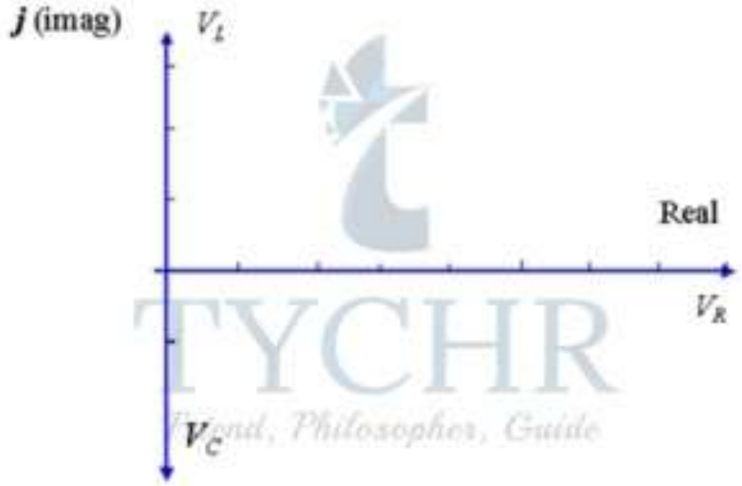
APPLICATIONS OF COMPLEX NUMBERS

REPRESENTING VOLTAGES IN COMPEX PLANE

Using the complex plane, we can represent voltages across resistors, capacitors and inductors.

The **voltage** across the **resistor** is regarded as a **real quantity**, while the voltage across an **inductor** is regarded as a **positive imaginary**

quantity, and across a **capacitor** we have a **negative imaginary quantity**. Our axes are as follows



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All AC waveforms have **sinusoidal** curves.

IMPEDANCE AND PHASE ANGLE

The **impedance** of a circuit is the **total effective resistance** to the flow of current by a combination of the elements of the circuit.

To find this total voltage, we cannot just add the voltages V_R, V_L and V_C.

Because V_L and V_C are considered to be imaginary quantities, we have:

Impedance $V_{RLC} = IZ$

$$Z = R + j(X_L - X_C)$$

$$|Z| = \sqrt{R^2 + (X_L - X_C)^2}$$

$$\tan\theta = (X_L - X_C)/R$$

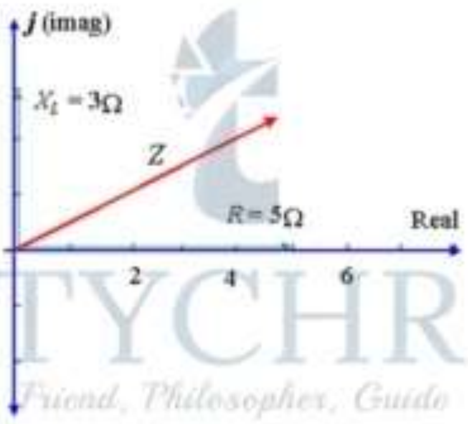
Example: A circuit has a resistance of 5Ω in series with a reactance across an inductor of 3Ω. Represent the impedance by a complex number, in polar form.

Solution: In this case, $X_L = 3\Omega$ and $X_C = 0$ so $X_L - X_C = 3\ \Omega$.

So in rectangular form, the impedance is written: $Z = 5 + 3j\ \Omega$

Using calculator, the magnitude of Z is given by: 5.83, and the angle θ is given by: 30.96°.

So, the voltage leads the current by 30.96°, as shown in the diagram.
Presenting Z as a complex number (in polar form), we have:
$Z = 5.83\angle 30.96^\circ\ \Omega$



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+91 9540653900